ON THE $KK$-THEORY OF STRONGLY SELF-ABSORBING $C^*$-ALGEBRAS

MARIUS DADARLAT AND WILHELM WINTER

Abstract. Let $D$ and $A$ be unital and separable $C^*$-algebras; let $D$ be strongly self-absorbing. It is known that any two unital $^*$-homomorphisms from $D$ to $A \otimes D$ are approximately unitarily equivalent. We show that, if $D$ is also $K_1$-injective, they are even asymptotically unitarily equivalent. This in particular implies that any unital endomorphism of $D$ is asymptotically inner. Moreover, the space of automorphisms of $D$ is compactly-contractible (in the point-norm topology) in the sense that for any compact Hausdorff space $X$, the set of homotopy classes $[X, \text{Aut}(D)]$ reduces to a point. The respective statement holds for the space of unital endomorphisms of $D$. As an application, we give a description of the Kasparov group $KK(D, A \otimes D)$ in terms of $^*$-homomorphisms and asymptotic unitary equivalence. Along the way, we show that the Kasparov group $KK(D, A \otimes D)$ is isomorphic to $K_0(A \otimes D)$.

0. Introduction

A unital and separable $C^*$-algebra $D \neq \mathbb{C}$ is strongly self-absorbing if there is an isomorphism $D \xrightarrow{\sim} D \otimes D$ which is approximately unitarily equivalent to the inclusion map $D \to D \otimes D$, $d \mapsto d \otimes 1_D$ ([14]). Strongly self-absorbing $C^*$-algebras are known to be simple and nuclear; moreover, they are either purely infinite or stably finite. The only known examples of strongly self-absorbing $C^*$-algebras are the UHF algebras of infinite type (i.e., every prime number that occurs in the respective supernatural number occurs with infinite multiplicity), the Cuntz algebras $O_2$ and $O_\infty$, the Jiang–Su algebra $Z$ and tensor products of $O_\infty$ with UHF algebras of infinite type, see [14]. All these examples are $K_1$-injective, i.e., the canonical map $\mathcal{U}(D)/\mathcal{U}_0(D) \to K_1(D)$ is injective.

It was observed in [14] that any two unital $^*$-homomorphisms $\sigma, \gamma : D \to A \otimes D$ are approximately unitarily equivalent, were $A$ is another unital and separable $C^*$-algebra. If $D$ is $K_1$-injective, the unitaries implementing the equivalence may even be chosen to

Date: May 29, 2007.
2000 Mathematics Subject Classification. 46L05, 47L40.
Key words and phrases. Strongly self-absorbing $C^*$-algebras, $KK$-theory, asymptotic unitary equivalence, continuous fields of $C^*$-algebras.
Supported by: The first named author was partially supported by NSF grant #DMS-0500693.
The second named author was supported by the DFG (SFB 478).
be homotopic to the unit. When $D$ is $O_2$, $O_\infty$, it was known that $\sigma$ and $\gamma$ are even asymptotically unitarily equivalent – i.e., they can be intertwined by a continuous path of unitaries, parametrized by a half-open interval. Up to this point, it was not clear whether the respective statement holds for the Jiang–Su algebra $Z$. Theorem 2.2 below provides an affirmative answer to this problem. Even more, we show that the path intertwining $\sigma$ and $\gamma$ may be chosen in the component of the unit.

We believe this result, albeit technical, is interesting in its own right, and that it will be a useful ingredient for the systematic further use of strongly self-absorbing $C^*$-algebras in Elliott’s program to classify nuclear $C^*$-algebras by $K$-theory data. In fact, this point of view is our main motivation for the study of strongly self-absorbing $C^*$-algebras; see [8], [10], [16], [17], [18] and [15] for already existing results in this direction.

For the time being, we use Theorem 2.2 to derive some consequences for the Kasparov groups of the form $KK(D, A \otimes D)$. More precisely, we show that all the elements of the Kasparov group $KK(D, A \otimes D)$ are of the form $[\varphi] - n[\iota]$ where $\varphi : D \rightarrow K \otimes A \otimes D$ is a *-homomorphism and $\iota : D \rightarrow A \otimes D$ is the inclusion $\iota(d) = 1_A \otimes d$ and $n \in \mathbb{N}$. Moreover, two non-zero *-homomorphisms $\varphi, \psi : D \rightarrow K \otimes A \otimes D$ with $\varphi(1_D) = \psi(1_D) = e$ have the same KK-theory class if and only if there is a unitary-valued continuous map $u : [0, 1) \rightarrow e(K \otimes A \otimes D)e, t \mapsto u_t$ such that $u_0 = e$ and $\lim_{t \rightarrow 1} \|u_t \varphi(d) u_t^* - \psi(d)\| = 0$ for all $d \in D$. In addition, we show that $KK_i(D, D \otimes A) \cong K_i(D \otimes A), i = 0, 1$.

One may note the similarity to the descriptions of $KK(O_\infty, O_\infty \otimes A)$ ([8],[10]) and $KK(C, C \otimes A)$. However, we do not require that $D$ satisfies the universal coefficient theorem (UCT) in KK-theory. In the same spirit, we characterize $O_2$ and the universal UHF algebra $Q$ using $K$-theoretic conditions, but without involving the UCT.

As another application of Theorem 2.2 (and the results of [7]), we prove in [4] an automatic trivialization result for continuous fields with strongly self-absorbing fibres over finite dimensional spaces.

The second named author would like to thank Eberhard Kirchberg for an inspiring conversation on the problem of proving Theorem 2.2.

1. STRONGLY SELF-ABSORBING $C^*$-ALGEBRAS

In this section we recall the notion of strongly self-absorbing $C^*$-algebras and some facts from [14].

1.1 Definition: Let $A$, $B$ be $C^*$-algebras and $\sigma, \gamma : A \rightarrow B$ be *-homomorphisms. Suppose that $B$ is unital.
We say that $\sigma$ and $\gamma$ are approximately unitarily equivalent, $\sigma \approx_u \gamma$, if there is a sequence $(u_n)_{n \in \mathbb{N}}$ of unitaries in $B$ such that
\[
\|u_n \sigma(a) u_n^* - \gamma(a)\| \xrightarrow{n \to \infty} 0
\]
for every $a \in A$. If all $u_n$ can be chosen to be in $U_0(B)$, the connected component of $1_B$ of the unitary group $U(B)$, then we say that $\sigma$ and $\gamma$ are strongly approximately unitarily equivalent, written $\sigma \approx_{su} \gamma$.

(ii) We say that $\sigma$ and $\gamma$ are asymptotically unitarily equivalent, $\sigma \approx_{uh} \gamma$, if there is a norm-continuous path $(u_t)_{t \in [0, \infty)}$ of unitaries in $B$ such that
\[
\|u_t \sigma(a) u_t^* - \gamma(a)\| \xrightarrow{t \to \infty} 0
\]
for every $a \in A$. If one can arrange that $u_0 = 1_B$ and hence $(u_t \in U_0(B))$ for all $t$, then we say that $\sigma$ and $\gamma$ are strongly asymptotically unitarily equivalent, written $\sigma \approx_{suh} \gamma$.

1.2 The concept of strongly self-absorbing $C^*$-algebras was formally introduced in [14, Definition 1.3]:

DEFINITION: A separable unital $C^*$-algebra $\mathcal{D}$ is strongly self-absorbing, if $\mathcal{D} \neq \mathbb{C}$ and there is an isomorphism $\varphi: \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ such that $\varphi \approx_u \text{id}_\mathcal{D} \otimes 1_\mathcal{D}$.

1.3 Recall [14, Corollary 1.12]:

PROPOSITION: Let $A$ and $\mathcal{D}$ be unital $C^*$-algebras, with $\mathcal{D}$ strongly self-absorbing. Then, any two unital *-homomorphisms $\sigma, \gamma: \mathcal{D} \to A \otimes \mathcal{D}$ are approximately unitarily equivalent. In particular, any two unital endomorphisms of $\mathcal{D}$ are approximately unitarily equivalent.

We note that the assumption that $A$ is separable which appears in the original statement of [14, Corollary 1.12] is not necessary and was not used in the proof.

1.4 LEMMA: Let $\mathcal{D}$ be a strongly self-absorbing $C^*$-algebra. Then there is a sequence of unitaries $(w_n)_{n \in \mathbb{N}}$ in the commutator subgroup of $U(\mathcal{D} \otimes \mathcal{D})$ such that for all $d \in \mathcal{D}$
\[
\|w_n(d \otimes 1_\mathcal{D})w_n^* - 1_\mathcal{D} \otimes d\| \to 0 \text{ as } n \to \infty.
\]

PROOF: Let $\mathcal{F} \subset \mathcal{D}$ be a finite normalized set and let $\varepsilon > 0$. By [14, Prop. 1.5] there is a unitary $u \in U(\mathcal{D} \otimes \mathcal{D})$ such that $\|u(d \otimes 1_\mathcal{D})u^* - 1_\mathcal{D} \otimes d\| < \varepsilon$ for all $d \in \mathcal{F}$. Let $\theta: \mathcal{D} \otimes \mathcal{D} \to \mathcal{D}$ be a *-isomorphism. Then $\|(\theta(u^*) \otimes 1_\mathcal{D})u(d \otimes 1_\mathcal{D})u^*(\theta(u) \otimes 1_\mathcal{D}) - 1_\mathcal{D} \otimes d\| < \varepsilon$ for all $d \in \mathcal{F}$. By Proposition 1.3 $\theta \otimes 1_\mathcal{D} \approx_u \text{id}_{\mathcal{D} \otimes \mathcal{D}}$ and so there is a unitary $v \in U(\mathcal{D} \otimes \mathcal{D})$ such that $\|\theta(u^*) \otimes 1_\mathcal{D} - vv^* u^*\| < \varepsilon$ and hence $\|\theta(u^*) \otimes 1_\mathcal{D})u - vv^* u\| < \varepsilon$. Setting $w = vv^* u$ we deduce that $\|w(d \otimes 1_\mathcal{D})w^* - 1_\mathcal{D} \otimes d\| < 3\varepsilon$ for all $d \in \mathcal{F}$.

1.5 REMARK: In the situation of Proposition 1.3, suppose that the commutator subgroup of $U(\mathcal{D})$ is contained in $U_0(\mathcal{D})$. This will happen for instance if $\mathcal{D}$ is assumed to be $K_1$-injective. Then one may choose the unitaries $(u_n)_{n \in \mathbb{N}}$ which implement the approximate
unitary equivalence between $\sigma$ and $\gamma$ to lie in $\mathcal{U}_0(A \otimes D)$. This follows from [14, (the proof of) Corollary 1.12], since the unitaries $(u_n)_{n \in \mathbb{N}}$ are essentially images of the unitaries $(w_n)_{n \in \mathbb{N}}$ of Lemma 1.4 under suitable unital $^\ast$-homomorphisms.

2. Asymptotic vs. approximate unitary equivalence

It is the aim of this section to establish a continuous version of Proposition 1.3.

2.1 Lemma: Let $\mathcal{D}$ be separable unital strongly self-absorbing $C^\ast$-algebra. For any finite subset $F \subset \mathcal{D}$ and $\varepsilon > 0$, there are a finite subset $G \subset \mathcal{D}$ and $\delta > 0$ such that the following holds:

If $A$ is another unital $C^\ast$-algebra and $\sigma : \mathcal{D} \to A \otimes \mathcal{D}$ is a unital $^\ast$-homomorphism, and if $w \in \mathcal{U}_0(A \otimes \mathcal{D})$ is a unitary satisfying

$$||[w, \sigma(d)]|| < \delta$$

for all $d \in G$, then there is a continuous path $(w_t)_{t \in [0,1]}$ of unitaries in $\mathcal{U}_0(A \otimes \mathcal{D})$ such that $w_0 = w, w_1 = 1_{A \otimes \mathcal{D}}$ and

$$||[w_t, \sigma(d)]|| < \varepsilon$$

for all $d \in F, t \in [0,1]$.

Proof: We may clearly assume that the elements of $F$ are normalized and that $\varepsilon < 1$. Let $u \in \mathcal{D} \otimes \mathcal{D}$ be a unitary satisfying

$$||u(d \otimes 1_\mathcal{D})u^* - 1_\mathcal{D} \otimes d|| < \frac{\varepsilon}{20}$$

for all $d \in F$. There exist $k \in \mathbb{N}$ and elements $s_1, \ldots, s_k, t_1, \ldots, t_k \in \mathcal{D}$ of norm at most one such that

$$||u - \sum_{i=1}^k s_i \otimes t_i|| < \frac{\varepsilon}{20}.$$  

Set

$$\delta := \frac{\varepsilon}{k \cdot 10}$$

and

$$G := \{s_1, \ldots, s_k\} \subset \mathcal{D}.$$  

Now let $w \in \mathcal{U}_0(A \otimes \mathcal{D})$ be a unitary as in the assertion of the lemma, i.e., $w$ satisfies

$$||[w, \sigma(s_i)]|| < \delta$$

for all $i = 1, \ldots, k$. We proceed to construct the path $(w_t)_{t \in [0,1]}$.

By [14, Remark 2.7] there is a unital $^\ast$-homomorphism

$$\varphi : A \otimes \mathcal{D} \otimes \mathcal{D} \to A \otimes \mathcal{D}$$
such that

\[ \| \varphi(a \otimes 1_D) - a \| < \frac{\varepsilon}{20} \]

for all \( a \in \sigma(F) \cup \{w\} \).

Since \( w \in U_0(A \otimes D) \), there is a path \((\tilde{w}_t)_{t \in \left[\frac{1}{2}, 1\right]}\) of unitaries in \( A \otimes D \) such that

\[ \tilde{w}_{\frac{1}{2}} = w \text{ and } \tilde{w}_1 = 1_{A \otimes D}. \]

For \( t \in \left[\frac{1}{2}, 1\right] \) define

\[ w_t := \varphi((\sigma \otimes \text{id}_D)(u)^*(\tilde{w}_t \otimes 1_D)(\sigma \otimes \text{id}_D)(u)) \in U(A \otimes D); \]

then \((w_t)_{t \in \left[\frac{1}{2}, 1\right]}\) is a continuous path of unitaries in \( A \otimes D \). For \( t \in \left[\frac{1}{2}, 1\right] \) and \( d \in F \) we have

\[
\begin{align*}
\|(w_t, \sigma(d))\| & = \|w_t \sigma(d) w_t^* - \sigma(d)\| \\
& < \|w_t \varphi(\sigma(d) \otimes 1_D)w_t^* - \varphi(\sigma(d) \otimes 1_D)\| + 2 \cdot \frac{\varepsilon}{20} \\
& \leq \|((\sigma \otimes \text{id}_D)(u)^*(\tilde{w}_t \otimes 1_D)((\sigma \otimes \text{id}_D)(u(d \otimes 1_D)u^*))((\tilde{w}_t^* \otimes 1_D) \\
& \quad - ((\sigma \otimes \text{id}_D)(d \otimes 1_D))\| + \frac{\varepsilon}{10} \\
& \leq \|((\sigma \otimes \text{id}_D)(u)^*(\tilde{w}_t \otimes 1_D)((\sigma \otimes \text{id}_D)(1_D \otimes d)(\tilde{w}_t^* \otimes 1_D) \\
& \quad - ((\sigma \otimes \text{id}_D)(d \otimes 1_D))\| + \frac{\varepsilon}{10} + \frac{\varepsilon}{20} \\
& = \|((\sigma \otimes \text{id}_D)(u)^*(1_D \otimes d)u - d \otimes 1_D)\| + \frac{\varepsilon}{10} + \frac{\varepsilon}{20} \\
& < \frac{\varepsilon}{20} + \frac{\varepsilon}{10} + \frac{\varepsilon}{20} \\
& < \frac{\varepsilon}{3};
\end{align*}
\]
where for the last equality we have used that the $\bar{w}_t$ are unitaries and that $\sigma$ is a unital $\ast$-homomorphism. Furthermore, we have

$$\|w_{\frac{1}{2}} - w\| \leq \|\varphi((\sigma \otimes \text{id}_D)(u)) - \varphi((\sigma \otimes \text{id}_D)(w))\| + \frac{\varepsilon}{20}$$

for all $t \in [0, \frac{1}{2})$, $d \in F$. We have now constructed a path $(w_t)_{t \in [\frac{1}{2}, 1]}$ to the whole interval $[0, 1]$ in the desired way: We have $\|w_{\frac{1}{2}} - 1\| < \frac{\varepsilon}{3} < 2$, whence $-1$ is not in the spectrum of $w_{\frac{1}{2}}$. By functional calculus, there is $a = a^* \in A \otimes D$ with $\|a\| < 1$ such that $w_{\frac{1}{2}} = \exp(\pi ia)$. For $t \in [0, \frac{1}{2})$ we may therefore define a continuous path of unitaries

$$w_t := (\exp(2\pi ita))w \in \mathcal{U}(A \otimes D).$$

It is clear that $w_0 = w$ and $w_t \rightarrow w_{\frac{1}{2}}$ as $t \rightarrow (\frac{1}{2})^-$, whence $(w_t)_{t \in [0, 1]}$ is a continuous path of unitaries in $A$ satisfying $w_0 = w$ and $w_1 = 1_A \otimes D$. Moreover, it is easy to see that

$$\|w_t - w\| < \frac{\varepsilon}{3}$$

for all $t \in [0, \frac{1}{2})$, whence

$$\|w_t, \sigma(d)\| < \|w_{\frac{1}{2}}, \sigma(d)\| + \frac{2}{3} \varepsilon < \varepsilon$$

for $t \in [0, \frac{1}{2})$, $d \in F$.

We have now constructed a path $(w_t)_{t \in [0, 1]} \subset \mathcal{U}(A)$ with the desired properties. 

**2.2 Theorem:** Let $A$ and $D$ be unital $C^*$-algebras, with $D$ separable, strongly self-absorbing and $K_1$-injective. Then, any two unital $\ast$-homomorphisms $\sigma, \gamma : D \rightarrow A \otimes D$ are strongly asymptotically unitarily equivalent. In particular, any two unital endomorphisms of $D$ are strongly asymptotically unitarily equivalent.
Proof: Note that the second statement follows from the first one with $A = D$, since $D \cong D \otimes D$ by assumption.

Let $A$ be a unital $C^*$-algebra such that $A \cong A \otimes D$ and let $\sigma, \gamma : D \to A$ be unital *-homomorphisms. We shall prove that $\sigma$ and $\gamma$ are strongly asymptotically unitarily equivalent. Choose an increasing sequence $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots$ of finite subsets of $D$ such that $\bigcup \mathcal{F}_n$ is a dense subset of $D$. Let $1 > \varepsilon_0 > \varepsilon_1 > \ldots$ be a decreasing sequence of strictly positive numbers converging to 0.

For each $n \in \mathbb{N}$, employ Lemma 2.1 (with $\mathcal{F}_n$ and $\varepsilon_n$ in place of $\mathcal{F}$ and $\varepsilon$) to obtain a finite subset $\mathcal{G}_n \subset D$ and $\delta_n > 0$. We may clearly assume that

$(10) \quad \mathcal{F}_n \subset \mathcal{G}_n \subset \mathcal{G}_{n+1}$

and that $\delta_{n+1} < \delta_n < \varepsilon_n$ for all $n \in \mathbb{N}$.

Since $\sigma$ and $\gamma$ are strongly approximately unitarily equivalent by Proposition 1.3 and Remark 1.5, there is a sequence of unitaries $(u_n)_{n \in \mathbb{N}} \subset U_0(A)$ such that

$(11) \quad \|u_n \sigma(d) u_n^* - \gamma(d)\| < \frac{\delta_n}{2}$

for all $d \in \mathcal{G}_n$ and $n \in \mathbb{N}$. Let us set

$w_n := u_{n+1}^* u_n, \quad n \in \mathbb{N}.$

Then $w_n \in U_0(A)$ and

$\|([w_n, \sigma(d)])\|$

$= \|w_n \sigma(d) w_n^* - \sigma(d)\|

\leq \|u_{n+1}^* u_n \sigma(d) u_n^* u_{n+1} - u_{n+1}^* \gamma(d) u_{n+1}\|

+ \|u_{n+1}^* \gamma(d) u_{n+1} - \sigma(d)\|

< \frac{\delta_n}{2} + \frac{\delta_{n+1}}{2}

< \delta_n$

for $d \in \mathcal{G}_n, \quad n \in \mathbb{N}$. Now by Lemma 2.1 (and the choice of the $\mathcal{G}_n$ and $\delta_n$), for each $n$ there is a continuous path $(w_{n,t})_{t \in [0,1]}$ of unitaries in $U_0(A)$ such that $w_{n,0} = w_n, \quad w_{n,1} = 1_A$ and

$(12) \quad \|[w_{n,t}, \sigma(d)]\| < \varepsilon_n$

for all $d \in \mathcal{F}_n, \quad t \in [0,1]$.

Next, define a path $(\bar{w}_t)_{t \in [0, \infty)}$ of unitaries in $U_0(A)$ by

$\bar{w}_t := u_{n+1} w_{n,t-n}$ if $t \in [n, n+1).$
We have that
\begin{equation}
\tilde{u}_n = u_{n+1} w_n = u_n
\end{equation}
and that
\[\tilde{u}_t \to u_{n+1}\]
as \(t \to n + 1\) from below, which implies that the path \((\tilde{u}_t)_{t \in [0, \infty)}\) is continuous in \(U_0(A)\). Furthermore, for \(t \in [n, n + 1)\) and \(d \in F_n\) we obtain
\begin{align*}
\|\tilde{u}_t \sigma(d) \tilde{u}_t^* - \gamma(d)\| &\leq \|u_{n+1} w_{n,t-n} \sigma(d) w_{n,t-n}^* u_{n+1}^* - \gamma(d)\| \\
&\overset{(12)}{<} \|u_{n+1} \sigma(d) u_{n+1}^* - \gamma(d)\| + \varepsilon_n \\
&\overset{(11),(10)}{<} \frac{\delta_{n+1}}{2} + \varepsilon_n \\
&\overset{(10)}{<} 2\varepsilon_n.
\end{align*}
Since the \(F_n\) are nested and the \(\varepsilon_n\) converge to 0, we have
\begin{equation}
\|\tilde{u}_t \sigma(d) \tilde{u}_t^* - \gamma(d)\| \xrightarrow{t \to \infty} 0
\end{equation}
for all \(d \in \bigcup_{n=0}^{\infty} F_n\); by continuity and since \(\bigcup_{n=0}^{\infty} F_n\) is dense in \(D\), we have (14) for all \(d \in D\). Since \(\tilde{u}_0 \in U_0(A)\) we may arrange that \(\tilde{u}_0 = 1_A\).

3. The group \(KK(D, A \otimes D)\) and some applications

3.1 For a separable \(C^*\)-algebra \(D\) we endow the group of automorphisms \(\text{Aut}(D)\) with the point-norm topology.

**Corollary:** Let \(D\) be a separable, unital, strongly self-absorbing and \(K_1\)-injective \(C^*\)-algebra. Then \([X, \text{Aut}(D)]\) reduces to a point for any compact Hausdorff space \(X\).

**Proof:** Let \(\varphi, \psi : X \to \text{Aut}(D)\) be continuous maps. We identify \(\varphi\) and \(\psi\) with unital \(^*\)-homomorphisms \(\varphi, \psi : D \to C(X) \otimes D\). By Theorem 2.2, \(\varphi\) is strongly asymptotically unitarily equivalent to \(\psi\). This gives a homotopy between the two maps \(\varphi, \psi : X \to \text{Aut}(D)\).

**Remark:** The conclusion of Corollary 3.1 was known before for \(D\) a UHF algebra of infinite type and \(X\) a CW complex by [13], for \(D = O_2\) by [8] and [10], and for \(D = O_{\infty}\) by [2]. It is new for the Jiang–Su algebra.

3.3 For unital \(C^*\)-algebras \(D\) and \(B\) we denote by \([D, B]\) the set of homotopy classes of unital \(^*\)-homomorphisms from \(D\) to \(B\). By a similar argument as above we also have the following corollary.
Corollary: Let $\mathcal{D}$ and $A$ be unital $C^*$-algebras. If $\mathcal{D}$ is separable, strongly self-absorbing and $K_1$-injective, then $[\mathcal{D}, A \otimes \mathcal{D}]$ reduces to a singleton.

3.4 For separable unital $C^*$-algebras $\mathcal{D}$ and $B$, let $\chi_i : KK_i(\mathcal{D}, B) \to KK_i(\mathbb{C}, B) \cong K_i(B)$, $i = 0, 1$ be the morphism of groups induced by the unital inclusion $\nu : \mathbb{C} \to \mathcal{D}$.

Theorem: Let $\mathcal{D}$ be a unital, separable and strongly self-absorbing $C^*$-algebra. Then for any separable $C^*$-algebra $A$, the map $\chi_i : KK_i(\mathcal{D}, A \otimes \mathcal{D}) \to K_i(A \otimes \mathcal{D})$ is bijective, for $i = 0, 1$. In particular both groups $KK_i(\mathcal{D}, A \otimes \mathcal{D})$ are countable and discrete with respect to their natural topology.

Proof: Since $\mathcal{D}$ is $KK$-equivalent to $\mathcal{D} \otimes O_\infty$, we may assume that $\mathcal{D}$ is purely infinite and in particular $K_1$-injective by [11, Prop. 4.1.4]. Let $C_\nu \mathcal{D}$ denote the mapping cone $C^*$-algebra of $\nu$. By [3, Cor. 3.10], there is a bijection $[\mathcal{D}, A \otimes \mathcal{D}] \to KK(C_\nu \mathcal{D}, SA \otimes \mathcal{D})$ and hence $KK(C_\nu \mathcal{D}, SA \otimes \mathcal{D}) = 0$ for all separable and unital $C^*$-algebras $A$ as a consequence of Corollary 3.3. Since $KK(C_\nu \mathcal{D}, A \otimes \mathcal{D})$ is isomorphic to $KK(C_\nu \mathcal{D}, S^2 A \otimes \mathcal{D})$ by Bott periodicity and the latter group injects in $KK(C_\nu \mathcal{D}, SC(\mathbb{T}) \otimes A \otimes \mathcal{D}) = 0$, we have that $KK_i(C_\nu \mathcal{D}, A \otimes \mathcal{D}) = 0$ for all unital and separable $C^*$-algebras $A$ and $i = 0, 1$. Since $KK(C_\nu \mathcal{D}, \mathcal{D} \otimes A)$ is a subgroup of $KK_i(C_\nu \mathcal{D}, \mathcal{D} \otimes A) = 0$ (where $\tilde{A}$ is the unitization of $A$) we see that $KK_i(C_\nu \mathcal{D}, \mathcal{D} \otimes A) = 0$ for all separable $C^*$-algebras $A$. Using the Puppe exact sequence, where $\chi_i = \nu^*$,

$$KK_{i+1}(C_\nu \mathcal{D}, A \otimes \mathcal{D}) \to KK_i(\mathcal{D}, A \otimes \mathcal{D}) \xrightarrow{\chi_i} KK_i(\mathbb{C}, A \otimes \mathcal{D}) \to KK_i(C_\nu \mathcal{D}, A \otimes \mathcal{D})$$

we conclude that $\chi_i$ is an isomorphism, $i = 0, 1$. The map $\chi_i = \nu^*$ is continuous since it is given by the Kasparov product with a fixed element (we refer the reader to [12], [9] or [1] for a background on the topology of the Kasparov groups). Since the topology of $K_i$ is discrete and $\chi_i$ is injective, it follows that the topology of $KK_i(\mathcal{D}, A \otimes \mathcal{D})$ is also discrete. The countability of $KK_i(\mathcal{D}, A \otimes \mathcal{D})$ follows from that of $K_i(A \otimes \mathcal{D})$, as $A \otimes \mathcal{D}$ is separable.

3.5 Remark: In contrast to Theorem 3.4, if $\mathcal{D}$ is the universal UHF algebra, then $KK(\mathcal{D}, \mathbb{C}) \cong \text{Ext}(\mathbb{Q}, \mathbb{Z}) \cong \mathbb{Q}^\mathbb{N}$ has the power of the continuum [6, p. 221].

3.6 Let $\mathcal{D}$ and $A$ be as in Theorem 3.4 and assume in addition that $\mathcal{D}$ is $K_1$-injective and $A$ is unital. Let $\iota : \mathcal{D} \to A \otimes \mathcal{D}$ be defined by $\iota(d) = 1_A \otimes d$.

Corollary: If $e \in \mathcal{K} \otimes A \otimes \mathcal{D}$ is a projection, and $\varphi, \psi : \mathcal{D} \to e(\mathcal{K} \otimes A \otimes \mathcal{D})e$ are two unital *-homomorphisms, then $\varphi \approx_{\text{suh}} \psi$ and hence $[\varphi] = [\psi] \in KK(\mathcal{D}, A \otimes \mathcal{D})$. Moreover:

$$KK(\mathcal{D}, A \otimes \mathcal{D}) = \{[\varphi] - n[\iota] | \varphi : \mathcal{D} \to \mathcal{K} \otimes A \otimes \mathcal{D} \text{ is a } *\text{-homomorphism, } n \in \mathbb{N}\}.$$
4. Characterizing $O_2$ and the Universal UHF Algebra

In the remainder of the paper we give characterizations for the Cuntz algebra $O_2$ and for the universal UHF-algebra which do not require the UCT. The latter result is a variation of a theorem of Effros and Rosenberg [5]. The results of this section do not depend on those of Section 2.

4.1 Proposition: Let $D$ be a separable unital strongly self-absorbing $C^*$-algebra. If $[1_D] = 0$ in $K_0(D)$, then $D \cong O_2$.

Proof: Since $D$ must be nuclear (see [14]), $D$ embeds unitally in $O_2$ by Kirchberg’s theorem. $D$ is not stably finite since $[1_D] = 0$. By the dichotomy of [14, Thm. 1.7] $D$ must be purely infinite. Since $[1_D] = 0$ in $K_0(D)$, there is a unital embedding $O_2 \to D$, see [11, Prop. 4.2.3]. We conclude that $D$ is isomorphic to $O_2$ by [14, Prop. 5.12].

4.2 Proposition: Let $D$, $A$ be separable, unital, strongly self-absorbing $C^*$-algebras. Suppose that for any finite subset $F$ of $D$ and any $\varepsilon > 0$ there is a u.c.p. map $\varphi : D \to A$ such that $\|\varphi(cd) - \varphi(c)\varphi(d)\| < \varepsilon$ for all $c, d \in F$. Then $A \cong A \otimes D$.

Proof: By [14, Thm. 2.2] it suffices to show that for any given finite subsets $F$ of $D$, $G$ of $A$ and any $\varepsilon > 0$ there is a u.c.p. map $\Phi : D \to A$ such that (i) $\|\Phi(cd) - \Phi(c)\Phi(d)\| < \varepsilon$ for all $c, d \in F$ and (ii) $\|\Phi(d), a\| < \varepsilon$ for all $d \in F$ and $a \in G$. We may assume that $\|d\| \leq 1$ for all $d \in F$. Since $A$ is strongly self-absorbing, by [14, Prop. 1.10] there is a unital $^*$-homomorphism $\gamma : A \otimes A \to A$ such that $\|\gamma(a \otimes 1_A) - a\| < \varepsilon/2$ for all $a \in G$. On the other hand, by assumption there is a u.c.p. map $\varphi : D \to A$ such that $\|\varphi(cd) - \varphi(c)\varphi(d)\| < \varepsilon$ for all $c, d \in F$. Let us define a u.c.p. map $\Phi : D \to A$ by $\Phi(d) = \gamma(1_A \otimes \varphi(d))$. It is clear that $\Phi$ satisfies (i) since $\gamma$ is a $^*$-homomorphism. To conclude the proof we check
now that $\Phi$ also satisfies (ii). Let $d \in \mathcal{F}$ and $a \in \mathcal{G}$. Then

$$
\|\Phi(d), a\| \\
\leq \|\Phi(d), a - \gamma(a \otimes 1_A)\| + \|\Phi(d), \gamma(a \otimes 1_A)\| \\
\leq 2\|\Phi(d)\| \|a - \gamma(a \otimes 1_A)\| + \|\gamma(1_A \otimes \varphi(d)), \gamma(a \otimes 1_A)\| \\
< 2\varepsilon/2 + 0 = \varepsilon.
$$

4.3 Proposition: Let $\mathcal{D}$ be a separable, unital, strongly self-absorbing $C^*$-algebra. Suppose that $\mathcal{D}$ is quasidiagonal, it has cancellation of projections and that $[1_D] \in nK_0(\mathcal{D})^+$ for all $n \geq 1$. Then $\mathcal{D}$ is isomorphic to the universal UHF algebra $\mathcal{Q}$ with $K_0(\mathcal{Q}) \cong \mathbb{Q}$.

Proof: Since $\mathcal{D}$ is separable unital and quasidiagonal, there is a unital $*$-representation $\pi : \mathcal{D} \to B(H)$ on a separable Hilbert space $H$ and a sequence of nonzero projections $p_n \in B(H)$ of finite rank $k(n)$ such that $\lim_{n \to \infty} \|[p_n, \pi(d)]\| = 0$ for all $d \in \mathcal{D}$. Then the sequence of u.c.p. maps $\varphi_n : \mathcal{D} \to \pi_n B(H) p_n \cong M_{k(n)}(\mathbb{C}) \subset \mathcal{Q}$ is asymptotically multiplicative, i.e $\lim_{n \to \infty} \|\varphi_n(cd) - \varphi_n(c)\varphi_n(d)\| = 0$ for all $c, d \in \mathcal{D}$. Therefore $\mathcal{Q} \cong \mathcal{Q} \otimes \mathcal{D}$ by Proposition 4.2.

In the second part of the proof we show that $\mathcal{D} \cong \mathcal{Q} \otimes \mathcal{D}$. Let $E_n : \mathcal{Q} \to M_{n!}(\mathbb{C}) \subset \mathcal{Q}$ be a conditional expectation onto $M_{n!}(\mathbb{C})$. Then $\lim_{n \to \infty} \|E_n(a) - a\| = 0$ for all $a \in \mathcal{Q}$.

By assumption, for each $n$ there is a projection $e$ in $\mathcal{D} \otimes M_{n!}(\mathbb{C})$ (for some $m$) such that $n!\{e\} = [1_D]$ in $K_0(\mathcal{D})$. Let $\varphi : M_{n!}(\mathbb{C}) \to M_{n!}(\mathbb{C}) \otimes e(\mathcal{D} \otimes M_{m!}(\mathbb{C}))$ be defined by $\varphi(b) = b \otimes e$. Since $\mathcal{D}$ has cancellation of projections and since $n!\{e\} = [1_D]$, there is a partial isometry $v \in M_{n!}(\mathbb{C}) \otimes \mathcal{D} \otimes M_{m!}(\mathbb{C})$ such that $v^*v = 1_M \otimes e$ and $vv^* = e$. Therefore $b \mapsto v \varphi(b) v^*$ gives a unital embedding of $M_{n!}(\mathbb{C})$ into $\mathcal{D}$. Finally, $\psi_n(a) = v (\varphi \circ E_n(a)) v^*$ defines a sequence of asymptotically multiplicative u.c.p. maps $\mathcal{Q} \to \mathcal{D}$. Therefore $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{Q}$ by Proposition 4.2.

4.4 Remark: Let $\mathcal{D}$ be a separable, unital, strongly self-absorbing and quasidiagonal $C^*$-algebra. Then $\mathcal{D} \otimes \mathcal{Q} \cong \mathcal{D}$ by the first part of the proof of Proposition 4.3. In particular $K_1(\mathcal{D}) \otimes \mathcal{Q} = 0$ and $K_0(\mathcal{D}) \otimes \mathcal{Q} \cong \mathcal{Q}$ by the Künneth formula (or by writing $\mathcal{Q}$ as an inductive limit of matrices).

References


Department of Mathematics, Purdue University, West Lafayette., IN 47907, USA

E-mail address: mdd@math.purdue.edu

Mathematisches Institut der Universität Münster, Einsteinstr. 62, D-48149 Münster, Germany

E-mail address: wwinter@math.uni-muenster.de