THE HOMOTOPY GROUPS OF THE AUTOMORPHISM GROUP OF KIRCHBERG ALGEBRAS

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Abstract. We compute the homotopy groups of the automorphism group of Kirchberg algebras. More generally, we calculate the homotopy classes $[X, \mathrm{Aut}(A)]$ for a Kirchberg algebra A and a path connected metrizable compact space X satisfying a natural continuity property.

1. Introduction

Kirchberg C*-algebras appear naturally in a variety of contexts [18]. They include the simple Cuntz-Krieger algebras associated to Markov chains [8], [5] or more generally the Ruelle algebras associated to hyperbolic homeomorphisms of compact spaces [17] as well as C*-algebras associated to boundary actions of certain groups and to a large class of groupoids [1], [14]. Remarkably, the action of any lattice of a real connected semisimple Lie group G without compact factors and with trivial centre on the Furstenberg boundary of G gives rise to a Kirchberg algebra [1]. The topological invariants of these algebras may reflect interesting geometric properties of the underlying dynamical systems. For instance the K-theory groups of the Cuntz-Krieger algebras $\mathcal{O}_A$ turned out to be exactly the invariants of flow equivalence for the matrix A discovered by R. Bowen and J. Franks [4].

The homotopy groups of the endomorphism space of the stable Cuntz-Krieger algebras were computed by Cuntz in [7]. In the early 80's, Cuntz asked for a computation of the homotopy groups of the automorphism groups of $\mathcal{O}_A$. Here we answer Cuntz's question by computing the homotopy groups of the automorphism group of an arbitrary Kirchberg algebra, see Corollary 5.10 and 5.11. The interest in this question has been renewed in view of our recent paper [9] which gives a simple KK-theoretical criterion for local triviality of a separable continuous field of Kirchberg algebras and which proves automatic local triviality of separable unital continuous fields of Cuntz algebras over finite dimensional compact Hausdorff spaces. These results reduce the study of many continuous fields to questions in

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homotopy theory. Indeed, if $\mathrm{Aut}(A)$ denotes the automorphism group of a Kirchberg algebra A endowed with the point-norm topology, the isomorphism classes of continuous fields over a compact Hausdorff space which are locally isomorphic to $C(X) \otimes A$ are parameterized by the homotopy classes $[X, B\mathrm{Aut}(A)]$ of maps into the classifying space for principal $\mathrm{Aut}(A)$-bundles. In view of the bijection $[SX, B\mathrm{Aut}(A)] \cong [X, \mathrm{Aut}(A)]/\pi_0(\mathrm{Aut}(A))$ (where SX is the unreduced suspension of X), the question of computing $[X, \mathrm{Aut}(A)]$ becomes quite interesting. Let us mention that this is a nontrivial question even when X is a point. While $\pi_0(\mathrm{Aut}(A))$ was computed in [16, Thm. 4.1.4] under the assumption that $K_1(A) = 0$, the general case has remained open.

Naturally our calculations rely on the fundamental work of Kirchberg [11], [12] and Phillips [16] on the classification of Kirchberg algebras. We were inspired by the work of Nistor on the homotopy theory of the automorphism group of AF-algebras [15] which illustrates the key role of the mapping cone $C_\nu A$ of the unital map $\nu : \mathbb{C} \to A$. Related ideas have appeared earlier in unpublished work of Skandalis [20] on the strong Ext-group.

Let us give an overview of the paper. With an eye on future applications we choose to work in a setup which is more general that what would be strictly required for this paper. Thus we consider homotopy classes of $*$-homomorphisms from a separable unital exact C*-algebra A to a separable unital properly infinite C*-algebra B. Section 2 recasts the classification results of Kirchberg and Phillips using the notion of nuclear absorbing $*$-homomorphisms. Section 3 investigates the natural action of $K_1(B)$ on $[A, B]_{una}$, the homotopy classes of unital absorbing $*$-homomorphisms from A to B. Inspired by [15] we establish a bijection $\chi : [A, B]_{una} \to KK_{nuc}(C_\nu A, SB)$, where $C_\nu A$ denotes the mapping cone of the unital map $\nu : \mathbb{C} \to A$. The proof that the map χ is bijective in our setting does not rely on the universal coefficient theorem for KK-theory, unlike [15]. Section 4 considers possible group structures on $[A, B]_{una}$. In particular it shows that if X is a path connected H'-space with nondegenerate base point x_0, then the map $\chi : [X, \mathrm{End}(A)^0] \to KK(C_\nu A, SC(X, x_0) \otimes A]$ is an isomorphism of abelian groups. Here $\mathrm{End}(A)^0$ stands for the path-component of id_A of the space of the unital endomorphisms $\mathrm{End}(A)$ of a unital Kirchberg algebra A. Section 5 determines the image of the natural map $[X, \mathrm{Aut}(A)] \to [X, \mathrm{End}(A)]$ and establishes a bijection $[X, \mathrm{Aut}(A)^0] \cong [X, \mathrm{End}(A)^0]$ under a suitable continuity condition on the pair X, A. This condition is automatically satisfied if X is locally contractible or if A is KK-semiprojective and hence it leads to the computation of the homotopy groups of $\mathrm{Aut}(A)$. Section 6 gives a description up to an extension of the (not necessarily abelian) group $[X, \mathrm{Aut}(A)]$.

The author is indebted to Larry Brown for providing him with a copy of [20].

2. KK-theory and $*$-homomorphisms

If a separable C*-algebra B has a full properly infinite projection, then $K_0(B) = KK(\mathbb{C}, B)$ has a simple realization due to Cuntz [6], see Proposition 2.1. In this section we reformulate results of Kirchberg and Phillips giving a similar realization for $KK_{nuc}(A, B)$ when A is a separable unital exact C*-algebra, see Proposition 2.8 and Theorem 2.9.

Two projections p, q in a C*-algebra A are Murray-Von Neumann equivalent, written $p \sim q$, if there is $v \in A$ such that $v^*v = p$ and $vv^* = q$. A projection $p \in A$ is called properly infinite if there are mutually orthogonal nonzero projections $p_1, p_2 \in A$ such that $p_1 + p_2 \leq p$ and $p \sim p_1 \sim p_2$. A unital C*-algebra A is called properly infinite if its unit 1_A is a properly infinite projection. A simple C*-algebra A is called purely infinite if A is not isomorphic to $\mathbb{C}$ and for any two positive nonzero elements $a, b \in A$, there is $c \in A$ such that $a = cbc^*$. A simple purely infinite nuclear separable C*-algebra is called a Kirchberg algebra [18]. Any nonunital Kirchberg algebra is of the form $A \otimes \mathcal{K}$ for some unital Kirchberg algebra A, see [23]. An element in a C*-algebra B is full if it is not contained in any proper two-sided closed ideal of B. A $*$-homomorphism $\varphi : A \to B$ is full if $\varphi(a)$ is full in B for any nonzero element $a \in A$. The following result is due to Cuntz [6].

Proposition 2.1. *Let B be a C*-algebra which contains a full properly infinite projection. For any $x \in K_0(B)$ there is a full properly infinite projection $p \in B$ such that $x = [p]$. If $p, q \in B$ are two full properly infinite projections such that $[p] = [q]$ then $p \sim q$. Moreover, if we also assume that B is unital and that both $1_B - p$ and $1_B - q$ are full and properly infinite, then $upu^* = q$ for some unitary $u \in U(B)$.*

Two $*$-homomorphisms $\varphi, \psi : A \to B$ are asymptotically unitarily equivalent, written $\varphi \approx_{uh} \psi$, if there is a norm continuous unitary valued map $t \mapsto u_t \in B^+$, $t \in [0, 1)$, such that $\lim_{t\to 1} \|u_t\varphi(a)u_t^* - \psi(a)\| = 0$ for all $a \in A$. By definition we set $B^+ = B$ if B is unital and $B^+ = \mathbb{C}1 + B$ otherwise. If $\varphi, \psi : A \to B$ are two maps we denote by $\varphi \oplus \psi : A \to M_2(B)$ the map $a \mapsto \begin{pmatrix} \varphi(a) & 0 \\ 0 & \psi(a) \end{pmatrix}$.

Definition 2.2. Let A and B be separable C*-algebras. A nuclear $*$-homomorphism $\varphi : A \to B \otimes \mathcal{K}$ is called *absorbing* if for any nuclear $*$-homomorphism $\psi : A \to B \otimes \mathcal{K}$ there is a nuclear $*$-homomorphism $\psi' : A \to B \otimes \mathcal{K}$ such that $\varphi \oplus 0 \approx_{uh} \psi \oplus \psi'$.

A $*$-homomorphism $\theta : A \to B$ is called $\mathcal{O}_2$*-factorable* if there are $*$-homomorphisms $\alpha : A \to \mathcal{O}_2$ and $\beta : \mathcal{O}_2 \to B$ such that $\theta = \beta\alpha$. Let us note that if A and B are nonzero, then θ is full if and only if α is injective and $\beta(1)$ is a full projection. If A is exact and B has a full properly infinite projection, then there is always a full $\mathcal{O}_2$-factorable

$*$-homomorphism from A to B. Indeed, any separable exact C*-algebra embeds in $\mathcal{O}_2$ by a theorem of Kirchberg [18, Thm. 6.3.11] and $\mathcal{O}_2$ admits a full embedding in B by [18, 4.1.4 and 4.2.3], since B has a full properly infinite projection. We shall need the following results of Kirchberg and Phillips, (see [18, Thm. 8.3.3] and [16, Thm. 4.1.1]). In order to ease notation, we will write $KK(\varphi)$ for the class of $*$-homomorphism $\varphi : A \to B$ in the group $KK_{nuc}(A, B)$ of [21].

Theorem 2.3 (Kirchberg's Classification Theorem)**.** *Let A be a separable exact unital C*-algebra and let B be a separable unital properly infinite C*-algebra. Fix a full $\mathcal{O}_2$-factorable $*$-homomorphism $\theta : A \to B$.*

(i) For every $\alpha \in KK_{nuc}(A, B)$ there is a full nuclear $*$-homomorphism $\varphi : A \to B$ such that $KK(\varphi) = \alpha$.

(ii) Let $\varphi, \psi : A \to B$ be two unital nuclear $*$-homomorphisms. Then $\varphi \oplus \theta \approx_{uh} \psi \oplus \theta$ if and only if $KK(\varphi) = KK(\psi)$.

(iii) Assume that B is purely infinite and simple and let $\varphi, \psi : A \to B$ be two unital nuclear injective $*$-homomorphisms. Then $\varphi \approx_{uh} \psi$ if and only if $KK(\varphi) = KK(\psi)$.

Theorem 2.4 (Phillips' Classification Theorem)**.** *Let A be a separable nuclear simple unital C*-algebra and let B be a separable unital C*-algebra that satisfies $B \cong B \otimes O_\infty$.*

(i) For every $\alpha \in KK(A, B)$ there is a full $*$-homomorphism $\varphi : A \to B \otimes \mathcal{K}$ such that $KK(\varphi) = \alpha$.

(ii) Let $\varphi, \psi : A \to B \otimes \mathcal{K}$ be two full $*$-homomorphisms. Then $\varphi \approx_{uh} \psi$ if and only if $KK(\varphi) = KK(\psi)$.

Lemma 2.5. *Let A and B be as in Theorem 2.3.*

(i) If $\theta, \theta' : A \to B \otimes \mathcal{K}$ are two full $\mathcal{O}_2$-factorable $*$-homomorphisms, then $\theta \approx_{uh} \theta'$.

(ii) Let $\varphi, \psi : A \to B \otimes \mathcal{K}$ be two nuclear $*$-homomorphisms such that $KK(\varphi) = KK(\psi)$. Then $\varphi \oplus \theta \approx_{uh} \psi \oplus \theta$ for any $*$-homomorphism $\theta : A \to B \otimes \mathcal{K}$ that is full and $\mathcal{O}_2$-factorable.

Proof. (i) Write $\theta = \beta\alpha$ and $\theta' = \beta'\alpha'$ where as in Definition 2.2 $\alpha, \alpha' : A \to \mathcal{O}_2$ are injective $*$-homomorphisms (which we may assume to be unital) and $\beta, \beta' : \mathcal{O}_2 \to B \otimes \mathcal{K}$ are full $*$-homomorphisms. We have $\alpha \approx_{uh} \alpha'$ by Theorem 2.3(iii) and $\beta \approx_{uh} \beta'$ by [3, Lemma 5.4]. It follows that $\theta \approx_{uh} \theta'$.

(ii) This is just a nonunital version of Theorem 2.3(ii). Set $\Phi = \varphi \oplus \theta$ and $\Psi = \psi \oplus \theta$. By (i) it suffices to prove that $\Phi \oplus \theta' \approx_{uh} \Psi \oplus \theta'$ for some full $\mathcal{O}_2$-factorable $*$-homomorphism $\theta' : A \to B \otimes \mathcal{K}$. Since $\Phi(1_A)$ and $\Psi(1_A)$ are full properly infinite projections in $B \otimes \mathcal{K}$ which have the same K_0-class, by Proposition 2.1 there is a unitary $w \in (B \otimes \mathcal{K})^+$ such that $\Phi(1_A) = w\Psi(1_A)w^*$. Set $e = \Phi(1_A)$ and fix a full $\mathcal{O}_2$-factorable $*$-homomorphism

$\theta' : A \to e(B \otimes \mathcal{K})e$. By Theorem 2.3(ii) applied to Φ and $w\Psi w^* : A \to e(B \otimes \mathcal{K})e$ we have $\Phi \oplus \theta' \approx_{uh} w\Psi w^* \oplus \theta'$ and hence $\Phi \oplus \theta' \approx_{uh} \Psi \oplus \theta'$. □

Proposition 2.6. *Let A and B be as in Theorem 2.3. A nuclear $*$-homomorphism $\varphi : A \to B \otimes \mathcal{K}$ is absorbing if and only if $\varphi \approx_{uh} \varphi \oplus \theta$ for some full $\mathcal{O}_2$-factorable $*$-homomorphism $\theta : A \to B \otimes \mathcal{K}$.*

Proof. Assume that φ is absorbing and fix a full $\mathcal{O}_2$-factorable $*$-homomorphism $\theta : A \to B \otimes \mathcal{K}$. By definition of absorption there is a nuclear $*$-homomorphism $\theta' : A \to B \otimes \mathcal{K}$ such that $\varphi \approx_{uh} \theta' \oplus \theta$. In particular, by using Lemma 2.5(i), we have

$$\varphi \oplus \theta \approx_{uh} \theta' \oplus \theta \oplus \theta \approx_{uh} \theta' \oplus \theta \approx_{uh} \varphi.$$

Conversely, assume that $\varphi \approx_{uh} \varphi \oplus \theta$ and let $\psi : A \to B \otimes \mathcal{K}$. By Theorem 2.3(i) there is a nuclear $*$-homomorphism $\psi' : A \to B \otimes \mathcal{K}$ such that $KK(\psi \oplus \psi') = 0$. We also have $KK(\theta) = 0$ as θ factors through $\mathcal{O}_2$. By Lemma 2.5(ii) $\psi \oplus \psi' \oplus \theta \approx_{uh} \theta \oplus \theta$ and $\theta \oplus \theta \approx_{uh} \theta$ by Lemma 2.5(i). Thus $\varphi \approx_{uh} \varphi \oplus \theta \approx_{uh} \varphi \oplus \psi \oplus \psi' \oplus \theta$. This shows that φ absorbs ψ. □

Corollary 2.7. *Let A, A' be separable exact unital C^*-algebras and let B, B' be separable unital properly infinite C^*-algebras. If $\varphi : A \to B \otimes \mathcal{K}$ is a nuclear absorbing $*$-homomorphism, then $\beta\,\varphi\,\alpha : A' \to B' \otimes \mathcal{K}$ is nuclear and absorbing whenever $\alpha : A' \to A$ is an injective $*$-homomorphism and $\beta : B \otimes \mathcal{K} \to B' \otimes \mathcal{K}$ is a $*$-homomorphism which maps full projections to full projections.*

Proof. This is an immediate consequence of Proposition 2.6. □

Proposition 2.8. *Let A be a separable exact unital C^*-algebra and let B be a separable unital properly infinite C^*-algebra. Let $\varphi : A \to B$ be a unital $*$-homomorphism. Then φ is automatically absorbing in each of the following cases:*

(i) A is nuclear purely infinite and simple.

(ii) A is nuclear and simple and $B \cong B \otimes \mathcal{O}_\infty$.

(iii) B is purely infinite and simple and $\varphi : A \to B$ is nuclear and injective.

Proof. (i) If A is nuclear purely infinite and simple, then id_A is absorbing by Theorem 2.3(iii) and so $\varphi = \varphi \circ \mathrm{id}_A$ is absorbing by Corollary 2.7. (ii) If A is nuclear and simple and $B \cong B \otimes \mathcal{O}_\infty$, then we see that φ is absorbing by applying Theorem 2.4(ii) to $\varphi \oplus \theta$ and φ. (iii) Similarly, if B is purely infinite and simple and if $\varphi : A \to B$ is nuclear and injective, then we verify that φ is absorbing by applying Theorem 2.3(iii) to $v(\varphi \oplus \theta)v^*$ and φ, where v is a unitary in $(B \otimes \mathcal{K})^+$ such that $v\big(1_B \oplus \theta(1_A)\big)v^* = 1_B \otimes e_{11}$. □

Our interest in absorbing $*$-homomorphisms stems from the following consequence of Theorem 2.3 which can be viewed as a far reaching generalization of Proposition 2.1. (Note that a projection p in a unital C*-algebra B is full and properly infinite if and only if the $*$-homomorphism $\varphi : \mathbb{C} \to B$, $\varphi(\lambda) = \lambda p$ is absorbing.)

Theorem 2.9. *Let A be a separable exact unital C*-algebra and let B be a separable unital properly infinite C*-algebra.*

(i) For any $\alpha \in KK_{nuc}(A,B)$ there is a nuclear absorbing $*$-homomorphism $\varphi : A \to B$ such that $KK(\varphi) = \alpha$. If $K_0(\alpha)[1_A] = [1_B]$ then we can arrange that $\varphi(1_A) = 1_B$.

(ii) If $\varphi, \psi : A \to B \otimes \mathcal{K}$ are two nuclear absorbing $*$-homomorphisms such that $KK(\varphi) = KK(\psi)$ then $\varphi \approx_{uh} \psi$. If $\varphi(1_A) = \psi(1_A) = e$, then we can arrange that the asymptotic unitary equivalence is induced by a continuous family of unitaries in $U(e(B \otimes \mathcal{K})e)$.

Proof. This follows from Proposition 2.1, Theorem 2.3, Lemma 2.5, Proposition 2.6 and from the observation that if $\varphi, \psi : A \to B \otimes \mathcal{K}$ are such that $\varphi(1_A) = \psi(1_A) = e$ and $\varphi \approx_{uh} \psi$, then there is a continuous map $t \mapsto u_t \in U(e(B \otimes \mathcal{K})e)$, $t \in [0,1)$, with the property that $\lim_{t\to 1} \|u_t\varphi(a)u_t^* - \psi(a)\| = 0$ for all $a \in A$ (as explained in the proof of [16, Thm. 4.1.4]). ☐

The set of nuclear absorbing $*$-homomorphisms from $A \otimes \mathcal{K}$ to $B \otimes \mathcal{K}$ is denoted $\mathrm{Hom}(A \otimes \mathcal{K}, B \otimes \mathcal{K})_{na}$. It is nonempty whenever A is separable and exact and B has a full properly infinite projection. The (possibly empty) set of nuclear unital absorbing $*$-homomorphisms from A to B is denoted $\mathrm{Hom}(A,B)_{una}$. These two spaces of homomorphisms are given the point-norm topology. The path components of $\mathrm{Hom}(A,B)_{una}$, also called homotopy classes, are denoted by $[A,B]_{una}$. The homotopy class of $\varphi \in \mathrm{Hom}(A,B)_{una}$ is denoted by $[\varphi]$. One defines similarly the set homotopy classes of nuclear absorbing $*$-homomorphism from $A \otimes \mathcal{K}$ to $B \otimes \mathcal{K}$ denoted by $[A \otimes \mathcal{K}, B \otimes \mathcal{K}]_{na}$. Since the unitary group of the multiplier algebra of a stable C*-algebra is path-connected one has the following immediate consequence of Theorem 2.9.

Proposition 2.10. *Let A be a separable exact unital C*-algebra and let B be a separable unital properly infinite C*-algebra. The map $\kappa : [A \otimes \mathcal{K}, B \otimes \mathcal{K}]_{na} \to KK_{nuc}(A,B)$, $\kappa[\varphi] = KK(\varphi)$ is an isomorphism of rings.*

The question of calculating $[A,B]_{una}$ is addressed in the next section.

3. The action of $K_1(B)$ on $[A,B]_{una}$

The main results of this section are Theorems 3.6 and 3.9. Throughout this section A is a separable exact unital C*-algebra and B is a separable unital properly infinite C*-algebra such that the natural map $U(B)/U(B)_0 \to K_1(B)$ is bijective. This is always the case if we assume that $B \cong B \otimes \mathcal{O}_\infty$ by [16, Lemma 2.1.7]. Here $U(B)_0$ denotes the path component of 1_B in the unitary group $U(B)$. Since $\mathrm{Hom}(A,B)_{una}$ can be the empty set we will have to assume that $\mathrm{Hom}(A,B)_{una}$ contains at least one element j which we use as a base-point. Let us note that it suffices to assume that there is a unital nuclear $*$-homomorphism $j: A \to B$. Indeed, by replacing j by $vj(-)v^* + \theta$ where $v \in B$ is a nonunitary isometry such that the projection $p = 1_B - vv^*$ is full and properly infinite and $\theta : A \to pBp$ is a full unital $\mathcal{O}_2$-factorable $*$-homomorphism, we may arrange that j is nuclear unital and absorbing.

Let us recall from [2] and [21] that if J is a separable C*-algebra, then each element of the Kasparov group $KK_{nuc}(A,J)$ can be represented by a strictly nuclear Cuntz pair, i.e. by a pair of strictly nuclear $*$-homomorphisms $\Phi, \Psi : A \to M(J \otimes \mathcal{K})$ such that $\Phi(a) - \Psi(a) \in J \otimes \mathcal{K}$ for all $a \in A$. The set of strictly nuclear Cuntz pairs is denoted by $\mathcal{E}_{nuc}(A,J)$. We reserve the notation $\langle \Phi, \Psi \rangle$ for the KK-theory class of the Cuntz pair (Φ,Ψ) in $KK_{nuc}(A,J)$.

Lemma 3.1. *If $\varphi, \psi \in \mathrm{Hom}(A,B)_{una}$, then $KK(\varphi) = KK(\psi)$ in $KK_{nuc}(A,B)$ if and only if $[v\varphi v^*] = [\psi]$ in $[A,B]_{una}$ for some $v \in U(B)$.*

Proof. Suppose that $[v\varphi v^*] = [\psi]$. Then $KK(\varphi) = KK(\psi)$ since the KK-functor is invariant under homotopy and unitary equivalence.

Conversely, suppose now that $KK(\varphi) = KK(\psi)$. By Theorem 2.9 there is a norm continuous unitary valued map $t \to u_t \in U(B)$, $t \in [0,1)$, such that $\lim_{t\to 1} \|u_t\varphi(d)u_t^* - \psi(d)\| = 0$ for all $d \in A$. This shows that $[v\varphi v^*] = [\psi]$ for $v = u_0$. □

Proposition 3.2. *Let A be a separable exact unital C*-algebra and let B be a separable unital properly infinite C*-algebra. For any $\varphi \in \mathrm{Hom}(A,B)_{una}$ and any $x \in KK_{nuc}(A,SB)$ there is a nuclear absorbing $*$-homomorphism $\Psi : A \to M_2(C[0,1] \otimes B)$, $\Psi = (\Psi_t)_{t\in[0,1]}$, such that $\Psi_0 = \Psi_1$, $x = \langle \Psi, \Psi_0 \rangle$ and Ψ_0 is of the form $\Psi_0 = \begin{pmatrix} \varphi_0 & 0 \\ 0 & 0 \end{pmatrix}$ where $\varphi_0 \in \mathrm{Hom}(A,B)_{una}$ and $[\varphi_0] = [\varphi] \in [A,B]_{una}$.*

Proof. Fix an element $x \in KK_{nuc}(A,SB)$. Using the split exact sequence

$$0 \longrightarrow KK_{nuc}(A,SB) \longrightarrow KK_{nuc}(A, C(S^1)\otimes B) \longrightarrow KK_{nuc}(A,B) \longrightarrow 0$$

we can regard x as an element of $KK_{nuc}(A, C(S^1)\otimes B)$. We identify S^1 with $[0,1]/0 \sim 1$. By Theorem 2.9 there is a nuclear absorbing $*$-homomorphism $\gamma : A \to C(S^1) \otimes B$, $\gamma = (\gamma_t)_{t\in[0,1]}$, $\gamma_0 = \gamma_1$ such that $KK(\gamma) = x$. We may also arrange that the projection $1_B - \gamma_0(1_A)$ is full and properly infinite. Since the image of x in $KK_{nuc}(A,B)$ vanishes, we deduce that $KK(\gamma_0)=0$. The $*$-homomorphisms

$$\alpha=\begin{pmatrix}\varphi & 0\\ 0 & \gamma_0\end{pmatrix},\ \beta=\begin{pmatrix}\varphi & 0\\ 0 & 0\end{pmatrix}: A\to M_2(B)$$

are nuclear, absorbing and $KK(\alpha)=KK(\beta)$. By Proposition 2.1 there is a unitary $w\in U_2(B)$ such that $w\alpha(1_A)w^*=\beta(1_A)$. By compressing both $w\alpha w^*$ and β by the projection $e=\begin{pmatrix}1_B & 0\\ 0 & 0\end{pmatrix}$ and setting $\psi=ew\alpha(-)w^*e$, we deduce that $\psi\in \mathrm{Hom}(A,B)_{una}$ and

$$KK(\psi)=KK(w\alpha w^*)=KK(\alpha)=KK(\varphi)+KK(\gamma_0)=KK(\varphi).$$

By Lemma 3.1 there is a unitary $u_0\in U(B)$ such that $[u_0\psi\, u_0^*]=[\varphi]$. Let us set $\varphi_0=u_0\psi\, u_0^*$ and $u=\begin{pmatrix}u_0 & 0\\ 0 & 1_B\end{pmatrix}\in U_2(B)$. We define $\Psi: A\to M_2(C[0,1]\otimes B)$, $\Psi=(\Psi_t)_{t\in[0,1]}$, $\Psi_1=\Psi_0$ by

$$\Psi_t=uw\begin{pmatrix}\varphi & 0\\ 0 & \gamma_t\end{pmatrix}w^*u^*.$$

Then $(\Psi,\Psi_0)\in\mathcal{E}_{nuc}(A,SB)$ and

$$\Psi_0=uw\begin{pmatrix}\varphi & 0\\ 0 & \gamma_0\end{pmatrix}w^*u^*=\begin{pmatrix}u_0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\psi & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}u_0^* & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}u_0\psi\, u_0^* & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix}\varphi_0 & 0\\ 0 & 0\end{pmatrix}.$$

We conclude the proof by observing that

$$\langle\Psi,\Psi_0\rangle=\langle\begin{pmatrix}\varphi & 0\\ 0 & \gamma\end{pmatrix},\begin{pmatrix}\varphi & 0\\ 0 & \gamma_0\end{pmatrix}\rangle=\langle\gamma,\gamma_0\rangle=KK(\gamma)-KK(\gamma_0)=x\in KK_{nuc}(A,SB).$$

□

Since $K_1(B)$ is isomorphic to $U(B)/U(B)_0$ we can define an action of $K_1(B)$ on $[A,B]_{una}$ by setting $[v]\cdot[\varphi]=[v\varphi\, v^*]$ for $v\in U(B)$ and $\varphi\in\mathrm{Hom}(A,B)_{una}$.

Having fixed a base point $j\in\mathrm{Hom}(A,B)_{una}$ we define a map $I: K_1(B)\to[A,B]_{una}$ by $I[v]=[v]\cdot[j]=[vjv^*]$. Let $\nu:\mathbb{C}\to A$ be defined by $\nu(\lambda)=\lambda 1_A$. We consider the following sequence of pointed sets and maps

$$KK_{nuc}(A,SB)\xrightarrow{Q}K_1(B)\xrightarrow{I}[A,B]_{una}\xrightarrow{T}KK_{nuc}(A,B)\xrightarrow{\nu^*}KK(\mathbb{C},B)$$

where T is defined by $T[\varphi]=KK(\varphi)-KK(j)$ and Q is the composition

$$KK_{nuc}(A,SB)\xrightarrow{\nu^*}KK(\mathbb{C},SB)\xrightarrow{\partial^{-1}}K_1(B)\ .$$

Here ∂ stands for the index map $\partial: K_1(B)\to K_0(SB)\cong KK(\mathbb{C},SB)$. The index map ∂ can be described as follows. Given $v\in U(B)$ there is a continuous path of unitaries

$\omega : [0,1] \to U_2(B)$, $t \mapsto \omega_t$, with endpoints $\omega_0 = \begin{pmatrix} v & 0 \\ 0 & v' \end{pmatrix}$ and $\omega_1 = 1_2$ where $v' \in U(B)$ is some unitary and 1_2 denotes the unit of $M_2(B)$. Then $p = \omega \begin{pmatrix} 1_B & 0 \\ 0 & 0 \end{pmatrix} \omega^*$ and $p_0 = \begin{pmatrix} 1_B & 0 \\ 0 & 0 \end{pmatrix}$ are projections in $M_2(C[0,1] \otimes B)$. If we let $\eta, \eta_0 : \mathbb{C} \to M_2(B)$ be defined by $\eta(\lambda) = \lambda p$ and $\eta_0(\lambda) = \lambda p_0$, then $(\eta, \eta_0) \in \mathcal{E}(\mathbb{C}, SB)$. It is well-known that (see [2]):

$$\partial[v] = \langle \eta, \eta_0 \rangle \in KK(\mathbb{C}, SB). \tag{1}$$

It will be useful for us to use the following description of $\partial^{-1} : KK(\mathbb{C}, SB) \to K_1(B)$. By Proposition 3.2 any given element $x \in KK(\mathbb{C}, SB)$ is represented by a Cuntz pair $(\eta, \eta_0) \in \mathcal{E}(\mathbb{C}, SB)$ where $\eta : \mathbb{C} \to M_2(C[0,1] \otimes B)$ is a $*$-homomorphism, $\eta = (\eta_t)_{t\in[0,1]}$, such that $\eta_1 = \eta_0$ and $\eta_0(1) = p_0 = \begin{pmatrix} 1_B & 0 \\ 0 & 0 \end{pmatrix}$. The equation $p_t = \eta_t(1)$ defines a continuous loop of projections in $M_2(B)$. Let $\omega : [0,1] \to U_2(B)$ be a continuous path of unitaries with $\omega_1 = 1_2$ and such that $p_t = \omega_t \, p_0 \, \omega_t^*$ for all $t \in [0,1]$. Then ω_0 must commute with p_0 and so $\omega_0 = \begin{pmatrix} v & 0 \\ 0 & v' \end{pmatrix}$ where v, v' are unitaries in $U(B)$. By the previous discussion $\partial[v] = \langle \eta, \eta_0 \rangle$ and hence

$$\partial^{-1}\langle \eta, \eta_0 \rangle = [v] \in K_1(B). \tag{2}$$

Lemma 3.3. *With notation as above,* $\mathrm{Image}(T) = \mathrm{Ker}(\nu^*)$.

Proof. If $\varphi \in \mathrm{Hom}(A,B)_{una}$, then after identifying $KK(\mathbb{C}, B)$ with $K_0(B)$ we have

$$\nu^* T[\varphi] = \nu^*(KK(\varphi) - KK(j)) = [\varphi(1_B)] - [j(1_B)] = 0.$$

Thus the image of T is contained in the kernel of ν^*.

To prove the reverse inclusion, let $x \in \mathrm{Ker}(\nu^*)$. The KK-element $x + KK(j)$ induces a map $K_0(A) \to K_0(B)$ which takes $[1_A]$ to $[1_B]$. By Theorem 2.9 there is $\varphi \in \mathrm{Hom}(A,B)_{una}$ such that $KK(\varphi) = x + KK(j)$ and hence $T[\varphi] = x$. $\square$

Let us note that Lemma 3.1 shows that if $\varphi, \psi \in \mathrm{Hom}(A,B)_{una}$, then $T[\varphi] = T[\psi]$ if and only if $[v\varphi\, v^*] = [\psi]$ for some $v \in U(B)$. Together with Lemma 3.3 this identifies the orbits of the action of $K_1(B)$ on $[A,B]_{una}$ with $\mathrm{Ker}(\nu^*) \subset KK_{nuc}(A,B)$. Next we describe the stabilizer groups of this action.

Proposition 3.4. *Let $v \in U(B)$ and $\varphi \in \mathrm{Hom}(A,B)_{una}$. Then $[v\varphi\, v^*] = [\varphi]$ in $[A,B]_{una}$ if and only if $[v] \in \mathrm{Image}(Q)$.*

Proof. Assume first that $[v\varphi\, v^*] = [\varphi]$. This implies that there is a nuclear unital $*$-homomorphism $\Phi : A \to C[0,1] \otimes B$, $\Phi = (\Phi_t)_{t\in[0,1]}$, such that $\Phi_0 = \varphi$ and $\Phi_1 = v\varphi\, v^*$. Let ω and v' be as in the description of the map ∂ given before (1). We define a $*$-homomorphism $\Psi : A \to M_2(C[0,1] \otimes B) \subset M(SB \otimes \mathcal{K})$, $\Psi = (\Psi)_{t\in[0,1]}$, by $\Psi_t = \omega_t \begin{pmatrix} \Phi_t & 0 \\ 0 & 0 \end{pmatrix} \omega_t^*$. Let us verify that $\Psi_1 = \Psi_0$. Indeed,

$$\Psi_0 = \omega_0\, \Phi_0\, \omega_0^* = \begin{pmatrix} v & 0 \\ 0 & v' \end{pmatrix} \begin{pmatrix} \varphi & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v^* & 0 \\ 0 & v'^{\,*} \end{pmatrix} = \begin{pmatrix} v\varphi\, v^* & 0 \\ 0 & 0 \end{pmatrix} = \Psi_1.$$

It follows that $(\Psi, \Psi_0) \in \mathcal{E}_{nuc}(A, SB)$ and hence $\langle \Psi, \Psi_0 \rangle \in KK_{nuc}(A, SB)$. We are going to verify that $\nu^*\langle \Psi, \Psi_0 \rangle = \partial[v]$ and so $[v] \in \mathrm{Image}(Q)$. Indeed by equation (1)

$$\nu^*\langle \Psi, \Psi_0 \rangle = \langle \Psi\nu, \Psi_0\nu \rangle = \langle \eta, \eta_0 \rangle = \partial[v].$$

Assume now that $v \in U(B)$ and $[v] = Q(x)$ for some $x \in KK_{nuc}(A, SB)$. We shall prove that $[\varphi] = [v\varphi\, v^*]$ for all $\varphi \in \mathrm{Hom}(A, B)_{una}$. Let $\Psi : A \to M_2(C[0,1] \otimes B)$ be a nuclear absorbing $*$-homomorphism given by Proposition 3.2 such that $\Psi_0 = \Psi_1$, $x = \langle \Psi, \Psi_0 \rangle$ and Ψ_0 is of the form $\Psi_0 = \begin{pmatrix} \varphi_0 & 0 \\ 0 & 0 \end{pmatrix}$ where $\varphi_0 \in \mathrm{Hom}(A, B)_{una}$ and $[\varphi_0] = [\varphi] \in [A, B]_{una}$. We note that $\nu^*\langle \Psi, \Psi_0 \rangle = \langle \lambda \mapsto \lambda p, \lambda \mapsto \lambda p_0 \rangle \in KK(\mathbb{C}, SB)$ where $p_t = \Psi_t(1_A)$, $t \in [0,1]$, is a continuous loop of projections in $M_2(B)$ and $p_1 = p_0 = \begin{pmatrix} 1_B & 0 \\ 0 & 0 \end{pmatrix}$.

Let $(\omega_t)_{t\in[0,1]}$ be a continuous path of unitaries in $U_2(B)$ such that $\omega_1 = 1_2$ and $p_t = \omega_t p_0 \omega_t^*$ for all $t \in [0,1]$. Since $p_0 = \omega_0 p_0 \omega_0^*$, $\omega_0 = \begin{pmatrix} v_0 & 0 \\ 0 & v_0' \end{pmatrix}$ for some unitaries $v_0, v_0' \in U(B)$. By equation (2), $Q(x) = \partial^{-1}\nu^*\langle \Psi, \Psi_0 \rangle = \partial^{-1}\langle p, p_0 \rangle = [v_0]$.

Let us observe that if we set $H_t = \omega_t^* \Psi_t \omega_t$, then $H = (H_t)_{t\in[0,1]}$ is a continuous path of nuclear absorbing $*$-homomorphisms from A to $M_2(B)$ such that $H_t(1_A) = p_0 = \begin{pmatrix} 1_B & 0 \\ 0 & 0 \end{pmatrix}$ for all $t \in [0,1]$. Consequently $h_t = p_0 H_t p_0$ is a continuous path in $\mathrm{Hom}(A, B)_{una}$ such that $h_0 = v_0^* \varphi_0\, v_0$ and $h_1 = \varphi_0$. In conclusion $[\varphi] = [\varphi_0] = [v_0^* \varphi_0\, v_0] = [v^* \varphi\, v]$ and hence $[\varphi] = [v\varphi\, v^*]$. $\square$

Corollary 3.5. *The natural map $[A, B]_{una} \to [A, B]_{un}$ is injective.*

Proof. Let $\varphi, \psi \in \mathrm{Hom}(A, B)_{una}$ be such that $[\varphi] = [\psi]$ in $[A, B]_{un}$. By Lemma 3.1 $[v\varphi v^*] = [\psi]$ in $[A, B]_{una}$ for some unitary $v \in U(B)$ and hence $[v\varphi v^*] = [\varphi]$ in $[A, B]_{un}$. By the first part of the proof of Proposition 3.4, $[v]$ must be in the image of Q (note that we worked there with a homotopy consisting of nuclear unital $*$-homomorphisms which were not necessarily absorbing). By applying the other implication of Proposition 3.4, we see that $[v\varphi v^*] = [\varphi]$ in $[A,B]_{una}$ and hence $[\varphi] = [\psi]$ in $[A,B]_{una}$. □

Proposition 3.4 shows that all the points of $[A,B]_{una}$ have the same stabilizer equal to the image of the map Q. Let us set

$$K_1(B)/\nu = K_1(B)/\mathrm{Image}(Q) = K_1(B)/\mathrm{Image}(\partial^{-1}\nu^*).$$

In conjunction with Lemma 3.3 we obtain:

Theorem 3.6. *The group $K_1(B)/\nu$ acts freely on $[A,B]_{una}$ and the orbit space is identified with $\mathrm{Ker}\big(\nu^* : KK_{nuc}(A,B) \to KK(\mathbb{C},B)\big)$ via the map $[\varphi] \mapsto KK(\varphi) - KK(j)$.*

Remark 3.7. There is an equally natural way to formulate Theorem 3.6. Let us set

$$KK_{nuc}(A,B)_u = \{\alpha \in KK_{nuc}(A,B) \,:\, K_0(\alpha)[1_A] = [1_B]\}.$$

Then the orbit space of the free action of $K_1(B)/\nu$ on $[A,B]_{una}$ is in bijection to $KK_{nuc}(A,B)_u$. This bijection is induced by the map $\kappa : [A,B]_{una} \to KK_{nuc}(A,B)_u$, $\kappa[\varphi] = KK(\varphi)$. We describe the situation by the following diagram, where we use a broken arrow to symbolize a group action.

$$K_1(B)/\nu \dashrightarrow [A,B]_{una} \xrightarrow{\kappa} KK_{nuc}(A,B)_u.$$

We shall apply Theorem 3.6 to identify $[A,B]_{una}$ with a more computable object. To this purpose we use the following map introduced by Nistor in [15]. Let $C_\nu A$ be the mapping cone of the unital map $\nu : \mathbb{C} \to A$:

$$C_\nu A = \{f \in C[0,1] \otimes A :\, f(0) \in \mathbb{C}1_A,\ f(1) = 0\}.$$

The mapping cone construction C_ν defines a functor from the category of unital C*-algebras and unital $*$-homomorphisms to the category of C*-algebras. If $\varphi : A \to B$ is a unital $*$-homomorphism, then $\Phi = C_\nu\varphi : C_\nu A \to C_\nu B$ is obtained as the restriction of $\mathrm{id}_{C[0,1]} \otimes \varphi$ to $C_\nu A$. If we set $J = C_\nu j$, then it is readily seen that $(\Phi, J) \in \mathcal{E}_{nuc}(C_\nu A, SB)$. Therefore we have a map $\chi : [A,B]_{una} \to KK_{nuc}(C_\nu A, SB)$ defined by

$$\chi[\varphi] = \langle C_\nu\varphi, C_\nu j\rangle = \langle \Phi, J\rangle.$$

By functoriality, the mapping cone construction preserves homotopies and so χ is well-defined. The short exact sequence

$$0 \longrightarrow SA \xrightarrow{i} C_\nu A \xrightarrow{\pi} \mathbb{C} \longrightarrow 0$$

induces an exact sequence of abelian groups where KK stands for KK_{nuc} ([2, Thm. 19.4.3]):

$$KK(A,SB) \xrightarrow{\nu^*} KK(\mathbb{C},SB) \xrightarrow{\pi^*} KK(C_\nu A,SB) \xrightarrow{i^*} KK(SA,SB) \xrightarrow{S\nu^*} KK(S\mathbb{C},SB)$$

In particular this shows that $KK(\mathbb{C}, SB)$ acts by translations on $KK_{nuc}(C_\nu A, SB)$ and identifies the orbit space with $\mathrm{Ker}(S\nu^*)$ and the stabilizer groups of this action with $\mathrm{Coker}(\nu^*)$.

The next Proposition shows that the map χ is equivariant modulo the identification of $K_1(B)$ with $KK(\mathbb{C}, SB)$.

Proposition 3.8. *If $v \in U(B)$ and $\varphi \in \mathrm{Hom}(A, B)_{una}$, then $\chi[v\varphi v^*] = \chi[\varphi] + \pi^*\partial[v]$.*

Proof. We have

$$\chi[v\varphi v^*] = \langle v\Phi v^*, J\rangle = \langle v\Phi v^*, vJv^*\rangle + \langle vJv^*, J\rangle = \langle \Phi, J\rangle + \langle vJv^*, J\rangle = \chi[\varphi] + \langle vJv^*, J\rangle.$$

Therefore it suffices to show that $\pi^*\partial[v] = \langle vJv^*, J\rangle$. It is convenient to denote by J both the map $\mathrm{id}_{C[0,1]} \otimes j : C[0,1] \otimes A \to C[0,1] \otimes B$ and its restriction to $C_\nu A$ which was the original definition of $J = C_\nu j$. Let ω be as in the definition of ∂ described just before equation (1). We define two homotopies of Cuntz pairs $(\gamma_s)_{s\in[0,1]}$ and $(\delta_s)_{s\in[0,1]}$ in $\mathcal{E}_{nuc}(C_\nu A, SB)$ as follows:

$$s \mapsto \gamma_s(f) = \left(\omega \begin{pmatrix} J(f(s\,\cdot)) & 0 \\ 0 & 0 \end{pmatrix} \omega^*, \begin{pmatrix} J(f(s\,\cdot)) & 0 \\ 0 & 0 \end{pmatrix}\right)$$

$$s \mapsto \delta_s(f) = \left(\omega_{(s\,\cdot)} \begin{pmatrix} J(f) & 0 \\ 0 & 0 \end{pmatrix} \omega^*_{(s\,\cdot)}, \begin{pmatrix} J(f) & 0 \\ 0 & 0 \end{pmatrix}\right).$$

One checks immediately that $\gamma_1 = \delta_1$ and

$$\gamma_0(f) = \left(\omega \begin{pmatrix} f(0)1_B & 0 \\ 0 & 0 \end{pmatrix} \omega^*, \begin{pmatrix} f(0)1_B & 0 \\ 0 & 0 \end{pmatrix}\right)$$

$$\delta_0(f) = \left(\begin{pmatrix} vJ(f)\,v^* & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} J(f) & 0 \\ 0 & 0 \end{pmatrix}\right).$$

Consequently,

$$\pi^*\partial[v] = \langle \gamma_0\rangle = \langle \delta_0\rangle = \langle vJ\,v^*, J\rangle.$$

□

One may find helpful to visualize the whole setup via the following commutative diagram where we let KK stand for KK_{nuc}.

$$\begin{tikzcd}
KK(A,SB) \arrow[r, "Q"] \arrow[d, equal] & K_1(B) \arrow[r, "I"] \arrow[d, "\partial"] & {[A,B]_{una}} \arrow[r, "T"] \arrow[d, "\chi"] & KK(A,B) \arrow[r, "\nu^*"] \arrow[d, "S"] & KK(\mathbb{C},B) \arrow[d, "S"] \\
KK(A,SB) \arrow[r, "\nu^*"] & KK(\mathbb{C},SB) \arrow[r, "\pi^*"] & KK(C_\nu A,SB) \arrow[r, "i^*"] & KK(SA,SB) \arrow[r, "S\nu^*"] & KK(S\mathbb{C},SB)
\end{tikzcd}$$

Here S stands for various suspension maps. The equality $\chi I = \pi^* \partial$ follows from Proposition 3.8 whereas the equality $ST = \imath^* \chi$ follows directly from the definitions of χ and T.

Theorem 3.9. *Let A be a separable exact unital C*-algebra and let B be a separable unital properly infinite C*-algebra such that the natural map $U(B)/U(B)_0 \to K_1(B)$ is bijective. Suppose that there is a nuclear unital $*$-homomorphism $j : A \to B$. Then the map $\chi : [A, B]_{una} \to KK_{nuc}(C_\nu A, SB)$ is a $K_1(B)$-equivariant bijection.*

Proof. We let $K_1(B)$ act by translation on $KK_{nuc}(C_\nu A, SB)$ by identifying $K_1(B)$ with $KK(\mathbb{C}, SB)$. As explained in the beginning of the section we may assume that j is also absorbing. The map χ is $K_1(B)$-equivariant by Proposition 3.8. The stabilizer group of each element $[\varphi]$ is the image of Q by Proposition 3.4 and this identifies with the stabilizer group of $\chi[\varphi]$ since ∂ maps one-to-one the image of Q onto the image of ν^* and the bottom sequence of the above commutative diagram is exact. We conclude the proof by noting that the suspension map $S : KK_{nuc}(A, B) \to KK_{nuc}(SA, SB)$ induces a bijection $\mathrm{Ker}(\nu^*) \to \mathrm{Ker}(S\nu^*)$ between the orbit spaces of the two actions. □

Let $[A, B]_u$ denote the homotopy classes of unital $*$-homomorphisms from A to B.

Corollary 3.10. *Let A be a separable nuclear simple unital C*-algebra and let B be a separable unital C*-algebra such that $B \cong B \otimes \mathcal{O}_\infty$. If there exists a unital $*$-homomorphism $j : A \to B$, then the map $\chi : [A, B]_u \to KK(C_\nu A, SB)$ is bijective.*

Proof. The map $U(B)/U(B)_0 \to K_1(B)$ is bijective by [16, Lemma 2.1.7] and $[A, B]_{una}$ coincides with $[A, B]_u$ by Proposition 2.8(ii). The statement follows now from Theorem 3.9. □

4. Group structure on homotopy classes

In general there is no natural algebraic structure on the homotopy classes $[A, B]$. However one can introduce a multiplicative structure provided that B is an 'H-space' in the category of C*-algebras. While we do not investigate this notion formally, we consider three natural classes of examples: (i) B is unital properly infinite and $[1_B] = 0$ in $K_0(B)$, (ii) $B = C(X, x_0) \otimes D$ where (X, x_0) is an H'-space (also called co-H-space) (iii) $B \cong B \otimes \mathcal{K}$.

Assume first that $[1_B] = 0$ in $K_0(B)$. Since A is exact and B is properly infinite, there is an absorbing unital $*$-homomorphism $\theta : A \to B$ which factors as $\theta = \beta\alpha$ where $\alpha : A \to \mathcal{O}_2$ and $\beta : \mathcal{O}_2 \to B$ are unital embeddings. Let $v_1, v_2 \in B$ be the images under β of the canonical generators s_1, s_2 of $\mathcal{O}_2$. One defines an $\mathcal{O}_2$-sum on $[A, B]_{una}$ by setting $[\varphi] +_\beta [\psi] = [\varphi \oplus_\beta \psi] = [v_1 \varphi v_1^* + v_2 \psi v_2^*]$. So far we only know that $+_\beta$ is a binary operation on $[A,B]_{una}$. However we have the following result which shows that $[A,B]_{una}$ is actually an abelian group whose isomorphism class does not depend on the choice of $\theta : A \to B$. Let us note that by Theorem 3.9 $[A,\mathcal{O}_2]_{una}$ reduces to a point, and hence $[\theta]$ depends only on $[\beta]$ and not on α.

Corollary 4.1. *Let A and B be as in Theorem 3.9 and assume that $[1_B]=0$ in $K_0(B)$. Then $\chi : [A,B]_{una} \to KK_{nuc}(C_\nu A, SB)$ is an isomorphism of groups. The addition on $[A,B]_{una}$ is given by the $\mathcal{O}_2$-sum $+_\beta$ and χ is defined using θ as a base-point.*

Proof. We first show that $[\theta]+_\beta[\theta]=[\theta]$. Let us note that

$$(\theta \oplus_\beta \theta)(a) = v_1\theta(a)v_1^* + v_2\theta(a)v_2^* = \beta(s_1\alpha(a)s_1^* + s_2\alpha(a)s_2^*) = \beta(\alpha \oplus_{id_{\mathcal{O}_2}} \alpha)(a).$$

By Theorem 3.9 $[A,\mathcal{O}_2]_{una}$ reduces to a point. Therefore $[\alpha \oplus_{id_{\mathcal{O}_2}} \alpha] = [\alpha]$, and hence

$$[\theta]+_\beta[\theta] = [\theta \oplus_\beta \theta] = [\beta(\alpha \oplus_{id_{\mathcal{O}_2}} \alpha)] = [\beta\alpha] = [\theta].$$

Next we show that $\chi([\varphi]+_\beta[\psi]) = \chi[\varphi]+\chi[\psi]$. Since χ is a bijection and $[\theta]+_\beta[\theta]=[\theta]$, this will imply that $\chi[\theta]=0$ and that the binary operation $+_\beta$ defines a group structure on $[A,B]_{una}$. In particular χ is an isomorphism of groups. We use the notation $\Phi = C_\nu\varphi$, $\Psi = C_\nu\psi$, $\Theta = C_\nu\theta$, with θ playing the role of the base-point so that $\chi[\varphi] = \langle \Phi, \Theta\rangle$. Then

$$\chi([\varphi]+_\beta[\psi]) = \chi(\varphi \oplus_\beta \psi) = \langle v_1\Phi v_1^* + v_2\Psi v_2^*, \Theta\rangle = \langle v_1\Phi v_1^* + v_2\Psi v_2^*, v_1\Theta v_1^* + v_2\Theta v_2^*\rangle$$

where the last equality follows since $v_1\theta v_1^* + v_2\theta v_2^*$ is homotopic to θ and χ is homotopy invariant. Using basic properties of the KK-groups we have

$$\langle v_1\Phi v_1^* + v_2\Psi v_2^*, v_1\Theta v_1^* + v_2\Theta v_2^*\rangle = \langle v_1\Phi v_1^*, v_1\Theta v_1^*\rangle + \langle v_2\Psi v_2^*, v_2\Theta v_2^*\rangle = \chi[\varphi]+\chi[\psi].$$

This completes the proof. $\square$

Remark 4.2. Corollary 4.1 is nontrivial even for $A=\mathcal{O}_2$ when it recovers a known isomorphism of groups: $[\mathcal{O}_2,B]_{una} = [\mathcal{O}_2,B]_u \cong K_1(B)$. In contrast $[A,\mathcal{O}_2]_{una} = [A,\mathcal{O}_2]_u = \{*\}$.

For C*-algebras A, B we endow the space $\mathrm{Hom}(A,B)$ of $*$-homomorphisms with the point-norm topology. If X is a compact Hausdorff space, then $\mathrm{Hom}(A, C(X)\otimes B)$ is homeomorphic to the space of continuous maps from X to $\mathrm{Hom}(A,B)$. We shall identify a $*$-homomorphism $\alpha \in \mathrm{Hom}(A, C(X)\otimes B)$ with the corresponding continuous map $X \to \mathrm{Hom}(A,B)$, $x \mapsto \alpha_x$, $\alpha_x(a) = \alpha(a)(x)$ for all $x\in X$ and $a \in A$. If $\alpha : A \to C(X)\otimes B$ is a $*$-homomorphism, let us denote by $\widetilde{\alpha} : C(X)\otimes A \to C(X)\otimes B$ its (unique) $C(X)$-linear extension and write $\widetilde{\alpha} \in \mathrm{Hom}_{C(X)}(C(X)\otimes A, C(X)\otimes B)$. We shall make without further comment the following identifications

$$\mathrm{Hom}_{C(X)}(C(X)\otimes A, C(X)\otimes B) \equiv \mathrm{Hom}(A, C(X)\otimes B) \equiv C(X, \mathrm{Hom}(A,B)). \tag{3}$$

Let us observe that if $\alpha, \beta \in \mathrm{Hom}(A, C(X) \otimes B)$ then $\widetilde{\alpha} \approx_{uh} \widetilde{\beta}$ if and only if $\alpha \approx_{uh} \beta$.

Remark 4.3. Here we review some basic facts from topology that will be needed in the sequel, see [22]. Let (X, x_0) and (Y, y_0) be pointed topological spaces. We denote by $[X, Y]$ the homotopy classes of continuous maps from X to Y and by $[(X, x_0), (Y, y_0)]$ the homotopy classes of base-point preserving continuous maps. Suppose that the point x_0 is nondegenerate, i.e. the inclusion map of x_0 in X is a cofibration. Then there is a natural action of $\pi_1(Y, y_0)$ on the homotopy classes $[(X, x_0), (Y, y_0)]$. If Y is path connected, then the orbit space of this action is identified with the base-point free homotopy classes $[X, Y]$.

Recall that a pointed space (Y, y_0) is an H-space if there is a continuous map (multiplication) $m : Y \times Y \to Y$ such that the maps $y \mapsto m(y, y_0)$ and $y \mapsto m(y_0, y)$ are homotopic to id_Y through base-point preserving maps $(Y, y_0) \to (Y, y_0)$. In particular $m(y_0, y_0) = y_0$. If (Y, y_0) is a an H-space, then the action of $\pi_1(Y, y_0)$ is trivial and so the natural map $[(X, x_0), (Y, y_0)] \to [X, Y]$ is bijective if Y is path connected.

If the multiplication m happens to be homotopy associative, then $[(X, x_0), (Y, y_0)]$ becomes a monoid. In general this monoid need not to be a group. However if X is a CW complex and if Y is path connected, then $[(X, x_0), (Y, y_0)] \cong [X, Y]$ is a group (see [22, Thm. 2.4, p. 462]).

A pointed space (X, x_0) is an H'-space if there is a continuous map (co-multiplication) $\mu : X \to X \vee X$ such that if $c : X \to X$ is the constant map that shrinks X to x_0 and if $p_1 = \mathrm{id}_X \vee c$, $p_2 = c \vee \mathrm{id}_X : X \vee X \to X$, then $p_1 \circ \mu$ and $p_2 \circ \mu$ are both homotopic to id_X through base-point preserving maps $(X, x_0) \to (X, x_0)$. We do not require the multiplication m or the co-multiplication μ to be homotopy (co-)associative or have an homotopy inverse. However we require the point x_0 to be nondegenerate. We also need to consider the inclusion maps $i_1, i_2 : X \to X \vee X$, $i_1(x) = x \vee x_0$ and $i_2(x) = x_0 \vee x$. They verify the equations $p_1 \circ i_1 = p_2 \circ i_2 = \mathrm{id}_X$.

If (X, x_0) is an H'-space and if (Y, y_0) is an H-space then we have two multiplications on $[(X, x_0), (Y, y_0)]$, one induced by m and the other induced by μ. It is well-known that these two operations coincide and they are commutative and associative (see [22, Thm. 5.21, p. 124]).

Fix a base-point $j \in \mathrm{Hom}(A, B)_{una}$. There is a map χ_X defined analogously to χ such that the following diagram is commutative:

$$\begin{tikzcd} {[(X, x_0), (\mathrm{Hom}(A, B)_{una}, j)]} \arrow[r, "\chi_X"] \arrow[d] & KK_{nuc}(C_\nu A, C(X, x_0) \otimes SB) \arrow[d] \\ {[A, C(X) \otimes B]_{una}} \arrow[r, "\chi"] & KK_{nuc}(C_\nu A, C(X) \otimes SB) \end{tikzcd}$$

This is verified by observing that if $\Phi = C_\nu(\varphi)$ and $J = C_\nu(j)$, then $\Phi(f) - J(f) \in C(X_0, x_0) \otimes SB$ for all $f \in C_\nu A$ provided that $\varphi_{x_0} = j$. Thus $\chi_X[\varphi] = \langle \Phi, J\rangle$ is well-defined.

Let us note that unlike the right hand vertical map, the left hand vertical map is not in general injective. This has to do with the distinction between the base-point preserving homotopy classes and the free homotopy classes as discussed in Remark 4.3.

If (X, x_0) is an H'-space, let $[\varphi] \star [\psi] = [(\varphi \vee \psi) \circ \mu]$ denote the induced multiplication on $[(X, x_0), (\mathrm{Hom}(A, B)_{una}, j)]$. The following two Propositions are proved similarly to [15, Lemma 5.1].

Proposition 4.4. *If (X, x_0) is an H'-space, then $\chi_X([\varphi] \star [\psi]) = \chi_X[\varphi] + \chi_X[\psi]$.*

Proof. We begin by observing that χ_X is a natural transformation. Thus if $h : (X, x_0) \to (Y, y_0)$ is a map of pointed spaces, which induces maps $h^* : C(Y, y_0) \to C(X, x_0)$, $h^* : KK_{nuc}(C_\nu A, C(Y, y_0) \otimes SB) \to KK_{nuc}(C_\nu A, C(X, x_0) \otimes SB)$ and $h^* : [(X, x_0), (\mathrm{Hom}(A, B)_{una}, j)] \to [(Y, y_0), (\mathrm{Hom}(A, B)_{una}, j)]$, then $h^* \circ \chi_Y = \chi_X \circ h^*$. In particular the co-multiplication $\mu : (X, x_0) \to (X \vee X, x_0)$ induces a commutative diagram

$$\begin{tikzcd}
{[(X \vee X, x_0), (\mathrm{Hom}(A, B)_{una}, j)]} \arrow[r, "\mu^*"] \arrow[d, "\chi_{X\vee X}"'] & {[(X, x_0), (\mathrm{Hom}(A, B)_{una}, j)]} \arrow[d, "\chi_X"] \\
KK_{nuc}(C_\nu A, C(X \vee X, x_0) \otimes SB) \arrow[r, "\mu^*"] & KK_{nuc}(C_\nu A, C(X, x_0) \otimes SB)
\end{tikzcd}$$

We assert that $\chi_{X\vee X}[\varphi \vee \psi] = p_1^* \chi_X[\varphi] + p_2^* \chi_X[\psi]$. This is verified by projecting both sides of the equation to $KK(C_\nu A, C(X, x_0) \otimes SB)$ via the maps i_1^* and i_2^*. Since $p_k \circ i_k = \mathrm{id}_X$, $k = 1, 2$, and $(\varphi \vee \psi) \circ i_1 = \varphi$, $(\varphi \vee \psi) \circ i_2 = \psi$, we see that both sides are mapped to $\chi_X[\varphi]$ by i_1^* and to $\chi_X[\psi]$ by i_2^*, so that they must be equal. Therefore

$$\chi_X([\varphi] \star [\psi]) = \chi_X[(\varphi \vee \psi) \circ \mu] = \chi_X \mu^*[\varphi \vee \psi] = \mu^* \chi_{X\vee X}[\varphi \vee \psi] =$$
$$\mu^*(p_1^* \chi_X[\varphi] + p_2^* \chi_X[\psi]) = (p_1 \circ \mu)^* \chi_X[\varphi] + (p_2 \circ \mu)^* \chi_X[\psi] = \chi_X[\varphi] + \chi_X[\psi],$$

since $p_1 \circ \mu$ and $p_2 \circ \mu$ are homotopic to id_X by the definition of H'-spaces. □

Similarly, if $\bar\chi_X : [(X, x_0), (\mathrm{Hom}(A \otimes \mathcal{K}, B \otimes \mathcal{K}), j)] \to KK(A, C(X, x_0) \otimes B)$ is defined by $\bar\chi_X[\varphi] = \langle \varphi, \varphi_{x_0}\rangle = \langle \varphi, j\rangle$, then we have

Proposition 4.5. *If (X, x_0) is an H'-space, then $\bar\chi_X([\varphi] \star [\psi]) = \bar\chi_X[\varphi] + \bar\chi_X[\psi]$.*

For a C*-algebra A we denote by $\mathrm{End}(A)$ the set of *nonzero* (*and unital if A is unital*) $*$-endomorphisms of A and by $\mathrm{End}(A)^0$ the path component of id_A in $\mathrm{End}(A)$.

Theorem 4.6. *Let A be a unital Kirchberg algebra and let X be a compact metrizable space. (a) There is a bijection*

$$\chi : [X, \mathrm{End}(A)\,] \to KK(C_\nu A, SC(X) \otimes A),$$

and an isomorphism of rings

$$\kappa : [X, \mathrm{End}(A \otimes \mathcal{K})\,] \to KK(A, C(X) \otimes A).$$

(b) If X is path connected and $x_0 \in X$, then there are bijections

$$\chi : [X, \mathrm{End}(A)^0\,] \to KK(C_\nu A, SC(X, x_0) \otimes A),$$

$$\bar{\chi} : [X, \mathrm{End}(A \otimes \mathcal{K})^0\,] \to KK(A, C(X, x_0) \otimes A).$$

If (X, x_0) is also an H'-space, then the maps χ and $\bar{\chi}$ are isomorphisms of abelian groups. The group structure induced by the co-multiplication of (X, x_0) coincides with the structure induced by the composition of endomorphisms.

Proof. (a) By Proposition 2.8, if A is unital, then $C(X, \mathrm{End}(A)) \cong \mathrm{Hom}(A, C(X) \otimes A)_{una}$. In other words, all unital $*$-homomorphisms $\varphi : A \to C(X) \otimes A$ are absorbing. A similar argument shows that $C(X, \mathrm{End}(A \otimes \mathcal{K})) \cong \mathrm{Hom}(A \otimes \mathcal{K}, C(X) \otimes A \otimes \mathcal{K})_{na}$. Indeed, as a consequence of Theorem 2.4, a $*$-homomorphism $\varphi : A \otimes \mathcal{K} \to C(X) \otimes A \otimes \mathcal{K}$ is absorbing if and only if $\varphi_x \neq 0$ for all $x \in X$. Therefore

$$[X, End(A)] \cong [A, C(X) \otimes A]_{una}, \quad [X, End(A \otimes \mathcal{K})] \cong [A \otimes \mathcal{K}, C(X) \otimes A \otimes \mathcal{K}]_{na}, \tag{4}$$

so that part (a) follows from Corollary 3.10 (with $j = j_A : A \to C(X) \otimes A$, $j_A(a) = 1_{C(X)} \otimes a$ for all $a \in A$) and Proposition 2.10.

(b) Since a continuous function maps a path component into a path component, we obtain from (4):

$$[X, End(A)^0] \cong \{[\varphi] \in [A, C(X) \otimes A]_{una} : [\varphi_{x_0}] = [\mathrm{id}_A] \in [A, A]_{una}\}.$$

$$[X, End(A \otimes \mathcal{K})^0] \cong \{[\varphi] \in [A \otimes \mathcal{K}, C(X) \otimes A \otimes \mathcal{K}]_{na} : [\varphi_{x_0}] = [\mathrm{id}_{A \otimes \mathcal{K}}] \in [A \otimes \mathcal{K}, A \otimes \mathcal{K}]_{na}\}.$$

For the first part of (b) we use part (a) and the commutative diagram of pointed sets

$$\begin{tikzcd}
{[X, \mathrm{End}(A)^0]} \arrow[r] \arrow[d, "\chi^0"] & {[A, C(X) \otimes A]_{una}} \arrow[r, "(\pi_{x_0})_*"] \arrow[d, "\chi"] & {[A, A]_{una}} \arrow[d, "\chi"] \\
KK(C_\nu A, SC(X, x_0) \otimes A) \arrow[r] & KK(C_\nu A, SC(X) \otimes A) \arrow[r, "(\pi_{x_0})_*"] & KK(C_\nu A, SA)
\end{tikzcd}$$

The vertical maps χ of this diagram are bijective by Corollary 3.10. Thus χ^0 maps bijectively $(\pi_{x_0})_*^{-1}([\mathrm{id}_A]) \cong [X, \mathrm{End}(A)^0]$ to $(\pi_{x_0})_*^{-1}(0) = KK(C_\nu A, SC(X, x_0) \otimes A)$.

The proof for the second part (b) is similar but we use instead the diagram

$$\begin{array}{ccccc} [X,\mathrm{End}(A\otimes\mathcal{K})^0] & \longrightarrow & [A\otimes\mathcal{K}, C(X)\otimes A\otimes\mathcal{K}]_{na} & \xrightarrow{(\pi_{x_0})_*} & [A\otimes\mathcal{K}, A\otimes\mathcal{K}]_{na} \\ \downarrow{\scriptstyle \kappa^0} & & \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \kappa} \\ \iota_A + KK(A, C(X,x_0)\otimes A) & \longrightarrow & KK(A, C(X)\otimes A) & \xrightarrow{(\pi_{x_0})_*} & KK(A,A) \end{array}$$

where the vertical maps $\kappa = KK(-)$ are bijective by Proposition 2.10 and ι_A is the KK-class of the $*$-homomorphism j_A. In this case the map $\bar{\chi}$ from the statement is defined by $\bar{\chi}[\varphi] = \kappa^0[\varphi] - \iota_A$.

Finally it remains to argue that the maps χ and $\bar{\chi}$ preserves the multiplicative structure induced by the co-multiplication of X. This follows from Propositions 4.4 and 4.5 since the homotopy classes $[X,\mathrm{End}(A)^0]$ and $[X,\mathrm{End}(A\otimes\mathcal{K})^0]$ coincide with the base-point preserving homotopy classes $[(X,x_0),(\mathrm{End}(A)^0,\mathrm{id}_A)]$ and $[(X,x_0),(\mathrm{End}(A\otimes\mathcal{K})^0,\mathrm{id}_{A\otimes\mathcal{K}})]$ as explained in Remark 4.3 since $\mathrm{End}(A)^0$ and $\mathrm{End}(A\otimes\mathcal{K})^0$ are path connected H-spaces. □

5. From endomorphisms to automorphisms

In this section we relate the homotopy theory of the space of endomorphisms which are KK-equivalences to the homotopy theory of the space of automorphisms (see Proposition 5.8). This leads to two of our main results: Theorems 5.9 and 6.3.

If $\widetilde{\alpha} : C(X)\otimes A \to C(X)\otimes B$ is a $C(X)$-linear $*$-homomorphism, we say that $\widetilde{\alpha}$ is full if its restriction to A, $\alpha : A \to C(X)\otimes B$, $\alpha(a) = \widetilde{\alpha}(1_{C(X)}\otimes a)$, is full in the usual sense.

Let A be a stable Kirchberg algebra and let X be a compact metrizable space. Let $t\mapsto \Phi_t \in \mathrm{End}_{C(X)}(C(X)\otimes A)$, $t\in[0,1)$, be a continuous path of full $C(X)$-linear $*$-endomorphisms and let $\Psi \in \mathrm{End}_{C(X)}(C(X)\otimes A)$ be a full $C(X)$-linear $*$-endomorphism.

Lemma 5.1. *Suppose that $KK(\Phi_0) = KK(\Psi)$. Then there is a continuous path of unitaries $t\mapsto u_t \in U((C(X)\otimes A)^+)_0$, $t\in[0,1)$, with the property that $\lim_{t\to 1}\|\Phi_t(a) - u_t\Psi(a)u_t^*\| = 0$ for all $a \in C(X)\otimes A$. T*

Proof. Two $C(X)$-linear $*$-homomorphisms $C(X)\otimes A \to C(X)\otimes A$ are asymptotically unitarily equivalent if and only if their restrictions to $A \cong 1_{C(X)}\otimes A$ are asymptotically unitarily equivalent. It is then clear that the conclusion of the lemma follows by applying [16, Lemma 4.1.2] to the restrictions of Φ_t and Ψ to A. □

Proposition 5.2. *Let A be a stable Kirchberg algebra and let X be a compact metrizable space. Let $\widetilde{\varphi}, \widetilde{\psi} \in \mathrm{End}_{C(X)}(C(X)\otimes A)$ be full $C(X)$-linear $*$-endomorphisms such* *that* $\widetilde{\varphi}\widetilde{\psi} \approx_{uh} \mathrm{id}_{C(X)\otimes A}$ *and* $\widetilde{\psi}\widetilde{\varphi} \approx_{uh} \mathrm{id}_{C(X)\otimes A}$. *Then there is an automorphism* $\Phi \in \mathrm{Aut}_{C(X)}(C(X)\otimes A)$ *such that* $\Phi \approx_{uh} \widetilde{\varphi}$.

Proof. As noted above, two $C(X)$-linear $*$-homomorphisms $\widetilde{\alpha}, \widetilde{\beta} : C(X)\otimes A \to C(X)\otimes A$ are asymptotically unitarily equivalent if and only if their restrictions to A, $\alpha = \widetilde{\alpha}|_A$ and $\beta = \widetilde{\beta}|_A$ have this property. By Theorem 2.9, this happens precisely when $KK(\alpha) = KK(\beta)$, provided that α and β are full. The proof of the proposition is essentially identical to the proof of [16, Theorem 4.2.1] except that one works with $C(X)$-linear $*$-homomorphisms and one replaces [16, Lemma 4.1.2] in the original arguments of [16] by Lemma 5.1. □

Definition 5.3. Let A be a separable C*-algebra and let X be a metrizable compact space. We say that the pair (A, X) is KK-continuous if for any point $x \in X$ there is a base of closed neighborhoods (V_n) of x such that the natural map $\varinjlim KK(A, C(V_n)\otimes A) \to KK(A, A)$ (induced by the evaluation map at x) is injective (and hence bijective).

Examples 5.4. Let us give some examples of KK-continuous pairs. A separable C*-algebra A is KK-semiprojective if the functor $KK(A, -)$ is continuous, i.e. for any inductive system $B_1 \to B_2 \to \ldots$ of separable C*-algebras, the map $\varinjlim KK(A, B_n) \to KK(A, \varinjlim B_n)$ is bijective. The class of KK-semiprojective C*-algebras includes the nuclear semiprojective C*-algebras (see [9]) and also the separable nuclear C*-algebras whose K-theory groups are finitely generated and which satisfy the Universal Coefficient Theorem in KK-theory (abbreviated UCT [19]) . It is clear from definition that if A is a KK-semiprojective C*-algebra then the pair (A, X) is KK-continuous for any compact metrizable space X. Also it is easy to check that if X is locally contractible, then the pair (A, X) is KK-continuous for any separable C*-algebra A.

Let us recall that the zero Čech-cohomology group with coefficients in a ring R, denoted by $\check{H}^0(X, R)$, consists of locally constant functions from X to R.

In the sequel we are going to use the notation $K_A(X) = KK(A, C(X)\otimes A)$ and $K_A(X, Y) = KK(A, C(X, Y)\otimes A)$ for Y a closed subspace of X. It is clear that $K_A(X, Y)$ extends to a generalized cohomology theory. The composition of the Kasparov product

$$K_A(X) \times K_A(X) \to K_A(X \times X)$$

with the restriction to diagonal map

$$K_A(X \times X) \to K_A(X)$$

defines a cup product on $K_A(X)$ which makes $K_A(X)$ into a ring. The multiplicative unit ι_A of $K_A(X)$ is given the KK-class of the $*$-homomorphism $j_A : A \to C(X)\otimes A$,

$j_A(a) = 1_{C(X)} \otimes a$ for all $a \in A$. Similarly one has a cup product

$$K_A(X,Y) \times K_A(X,Y') \to K_A(X, Y \cup Y'),$$

which is compatible with the cup product on $K_A(X)$.

Remark 5.5. One can identify Kasparov's RKK-theory group $RKK(X,A,B)$ introduced in [10] with $KK(A, C(X) \otimes B)$ via the natural restriction map. This gives an isomorphism of rings $RKK(X,A,A) \to K_A(X)$.

Proposition 5.6. *If the pair (A,X) is KK-continuous, then an element σ is invertible in the ring $K_A(X)$ if and only if $\sigma_x \in KK(A,A)^{-1}$ for all $x \in X$.*

Proof. For each $x \in X$, the evaluation map at x gives a split exact sequence $0 \to C(X,x) \otimes A \to C(X) \otimes A \to A \to 0$. The splitting map is obtained by regarding the elements of A as constant functions on X. Therefore we have a split exact sequence

$$0 \to K_A(X,x) \to K_A(X) \to K_A(x) = KK(A,A) \to 0.$$

For $\sigma \in K_A(X)$, let $\widehat{\sigma} : X \to KK(A,A)$ be the map $\widehat{\sigma}(x) = \sigma_x$. Since by assumption the map $\varinjlim K_A(V_n) \to KK(A,A)$ is injective, we see immediately that the map $\widehat{\sigma}$ is locally constant. Therefore we have a split exact sequence of rings

$$0 \longrightarrow K'_A(X) \longrightarrow K_A(X) \xrightarrow{\rho} \check{H}^0(X, KK(A,A)) \longrightarrow 0\ , \tag{5}$$

where by definition $\rho(\sigma) = \widehat{\sigma}$ and $K'_A(X) = \mathrm{Ker}(\rho)$. We are going to prove that $K'_A(X)$ is a nil ideal, i.e. if $\sigma \in K'_A(X)$ then $\sigma^m = 0$ for some $m \geq 1$. For this purpose fix $\sigma \in K'_A(X) = \bigcap_{x \in X} K_A(X,x) \subset K_A(X)$. For every $x \in X$, let (V_n) be a base of closed neighborhoods of x as in Definition 5.3. Since the map $\varinjlim K_A(V_n) \to K_A(x)$ is injective, by diagram chasing (or by Steenrod's five lemma)

$$\begin{tikzcd}
 & \varinjlim K_A(X,V_n) \arrow[r] \arrow[d] & K_A(X) \arrow[r] \arrow[d, equal] & \varinjlim K_A(V_n) \arrow[d] & \\
0 \arrow[r] & K_A(X,x) \arrow[r] & K_A(X) \arrow[r] & K_A(x) \arrow[r] & 0
\end{tikzcd}$$

we obtain that the natural map $\varinjlim K_A(X,V_n) \to K_A(X,x)$ is surjective for every $x \in X$. Using the compactness of X we find a cover of X by finitely many closed subsets $Y_1, \dots, Y_m$ and elements $\sigma_k \in K_A(X,Y_k)$ $k = 1, \dots, m$ such that each σ_k maps to σ under the map $K_A(X,Y_k) \to K_A(X)$. It follows that σ^m is equal to the image of the cup product $\sigma_1 \cdots \sigma_m \in K_A(X, Y_1 \cup \cdots \cup Y_m) = K_A(X,X) = 0$, and hence $\sigma^m = 0$. Since $K'_A(X)$ is a nil ideal, and since ρ admits a multiplicative splitting, an element $\sigma \in K_A(X)$ is invertible if and only if $\rho(\sigma) = \widehat{\sigma}$ is invertible. $\square$

Next we describe the range of the map $\mathrm{Aut}_{C(X)}(C(X)\otimes A)\to KK(A,C(X)\otimes A)$.

Proposition 5.7. *Let A be a Kirchberg algebra and let X be a compact metrizable space. Suppose that the pair (A,X) is KK-continuous. Let $\sigma\in KK(A,C(X)\otimes A)$ be such that $\sigma_x\in KK(A,A)^{-1}$ for all $x\in X$ and $K_0(\sigma)[1_A]=[1_A]$ if A is unital. Then there is an automorphism $\alpha\in\mathrm{Aut}_{C(X)}(C(X)\otimes A)$ such that $KK(\alpha|_A)=\sigma$.*

Proof. Let $\sigma'\in KK(A,C(X)\otimes A)$ be the multiplicative inverse of σ given by Proposition 5.6. By Theorem 2.9 we can lift σ and σ' to full (and unital if A is unital) $*$-homomorphisms $\varphi,\psi: A\to C(X)\otimes A$. Therefore

$$KK(\widetilde{\varphi}\,\psi)=KK(\widetilde{\psi}\,\varphi)=KK(j_A),$$

where $\widetilde{\varphi}$ and $\widetilde{\psi}$ denote the $C(X)$-linear extensions of φ and ψ. By Theorem 2.9 we have $\widetilde{\varphi}\psi\approx_{uh} j_A$ and $\widetilde{\psi}\varphi\approx_{uh} j_A$ and hence $\widetilde{\varphi}\widetilde{\psi}\approx_{uh}\mathrm{id}_{C(X)\otimes A}$ and $\widetilde{\psi}\widetilde{\varphi}\approx_{uh}\mathrm{id}_{C(X)\otimes A}$. If A is stable we apply Proposition 5.2 to find an automorphism $\Phi\in\mathrm{Aut}_{C(X)}(C(X)\otimes A)$ such that $\Phi\approx_{uh}\widetilde{\varphi}$. In particular it follows that $KK(\Phi|_A)=\sigma$. If A is unital we apply Proposition 5.2 for $\widetilde{\varphi}\otimes\mathrm{id}_{\mathcal{K}}$ and $\widetilde{\psi}\otimes\mathrm{id}_{\mathcal{K}}$ to find an automorphism $\Phi\in\mathrm{Aut}_{C(X)}(C(X)\otimes A\otimes\mathcal{K})$ such that $\Phi\approx_{uh}\widetilde{\varphi}\otimes\mathrm{id}_{\mathcal{K}}$. By Theorem 2.9 we may arrange that $\Phi(1_{C(X)\otimes A}\otimes e_{11})=\widetilde{\varphi}(1_{C(X)\otimes A}\otimes e_{11})=1_{C(X)\otimes A}\otimes e_{11}$ and hence the compression of Φ to the $(1,1)$-corner $C(X)\otimes A\otimes e_{11}$ of $C(X)\otimes A\otimes\mathcal{K}$ gives an automorphism $\alpha\in\mathrm{Aut}_{C(X)}(C(X)\otimes A)$ such that $KK(\alpha|_A)=KK(\varphi)=\sigma$. $\square$

Let us set $\mathrm{End}(A)^*=\{\gamma\in\mathrm{End}(A): KK(\gamma)\in KK(A,A)^{-1}\}$.

Proposition 5.8. *Let A be a Kirchberg algebra and let X be a compact metrizable space. Suppose that the pair (A,X) is KK-continuous. Then the natural map $[X,\mathrm{Aut}(A)]\to[X,\mathrm{End}(A)^*]$ is bijective.*

Proof. We assert that for any given continuous map $x\mapsto\varphi_x\in\mathrm{End}(A)^*$ defined on X there is a continuous maps $(x,t)\mapsto\Phi_{(x,t)}\in\mathrm{End}(A)^*$ defined on $X\times[0,1]$ such that $\Phi_{(x,0)}=\varphi_x$ and $\Phi_{(x,t)}\in\mathrm{Aut}(A)$ for all $x\in X$ and $t\in(0,1]$. The proposition is an immediate consequence of our assertion. Let us prove now the assertion. By Proposition 5.2 there is an automorphism $\alpha\in\mathrm{Aut}_{C(X)}(C(X)\otimes A)$ such that $KK(\alpha|_A)=KK(\varphi)$. By Theorem 2.9 there is continuous map $(0,1]\to U((C(X)\otimes A)^+)$, $t\mapsto u_t$, with the property that

$$\lim_{t\to 0}\|u_t\alpha(a)u_t^*-\varphi(a)\|=0,\ \text{for all}\quad a\in A.$$

The equation

$$\Phi_{(x,t)}=\begin{cases}\varphi_x, & \text{if } t=0,\\ u_t(x)\alpha_x u_t(x)^*, & \text{if } t\in(0,1],\end{cases}$$

defines a continuous map $\Phi : X \times [0,1] \to \mathrm{End}(A)^*$ which extends φ and such that $\Phi(X \times (0,1]) \subset \mathrm{Aut}(A)$. □

If A is a C*-algebra let us denote by $\mathrm{Aut}(A)^0$ the path component of id_A in the automorphism group $\mathrm{Aut}(A)$ of A. If the point x_0 is nondegenerate, then the set of basepoint-free homotopy classes $[X, \mathrm{Aut}(A)^0]$ coincide with the set of base-point preserving homotopy classes $[(X, x_0), (\mathrm{Aut}(A)^0, \mathrm{id}_A)]$. The group multiplication of $\mathrm{Aut}(A)^0$ induces a group structure on the set of homotopy classes $[X, \mathrm{Aut}(A)^0]$ which coincides with the alternate group structure induced in the case when (X, x_0) is an H'-space (see Remark 4.3).

Theorem 5.9. *Let A be a unital Kirchberg algebra, let X be a path connected compact metrizable space and let $x_0 \in X$. Suppose that the pair (A, X) is KK-continuous. Then there are bijections*

$$\chi : [\,X, \mathrm{Aut}(A)^0] \to KK(C_\nu A, SC(X, x_0) \otimes A),$$

$$\bar{\chi} : [\,X, \mathrm{Aut}(A \otimes \mathcal{K})^0] \to KK(A, C(X, x_0) \otimes A).$$

If (X, x_0) is an H'-space, then χ and $\bar{\chi}$ are isomorphisms of abelian groups.

Proof. The result follows from Theorem 4.6 and Proposition 5.8. □

Recall that the map $Q = \partial^{-1}\nu^*$ is given by the composition

$$KK(A, SA) \xrightarrow{\nu^*} KK(\mathbb{C}, SA) \xrightarrow{\partial^{-1}} K_1(A)\ .$$

Let us set

$$K_1(A)/\nu = K_1(A)/\mathrm{Image}\, Q.$$

From Theorem 5.9 and Remark 3.7 we deduce the following:

Corollary 5.10. *Let A be a unital Kirchberg algebra. There are group isomorphisms $\chi : \pi_n \mathrm{Aut}(A) \to KK(C_\nu A, S^{n+1}A)$ for $n \geq 1$ and there is an exact sequence of groups $1 \to K_1(A)/\nu \to \pi_0\mathrm{Aut}(A) \to KK(A, A)_u^{-1} \to 1$ for $n = 0$.*

Corollary 5.11. *Let A be a Kirchberg algebra. There are group isomorphisms $\kappa : \pi_0\mathrm{Aut}(A \otimes \mathcal{K}) \to KK(A, A)^{-1}$ and $\bar{\chi} : \pi_n \mathrm{Aut}(A \otimes \mathcal{K}) \to KK(A, S^n A)$ for $n \geq 1$.*

From Theorem 5.9 and Corollary 5.10 we obtain a bijection

$$[X, Aut(A)] \cong KK(C_\nu A, SC(X, x_0) \otimes A) \times KK(A, A)_u^{-1} \times K_1(A)/\nu.$$

The K-theory groups of $C_\nu A$ can be often computed using the exact sequence:

$$0 \longrightarrow K_1(A) \xrightarrow{i_*\partial} K_0(C_\nu A) \longrightarrow \mathbb{Z} \xrightarrow{\nu_*} K_0(A) \longrightarrow K_1(C_\nu A) \longrightarrow 0.$$

Thus if $K_1(A)=0$ then $K_1(C_\nu A)\cong K_0(A)/\mathbb{Z}[1_A]$ and $K_0(C_\nu A)\cong\{k\in\mathbb{Z}: k[1_A]=0\}$ so that $K_0(C_\nu A)$ is either isomorphic to $\mathbb{Z}$ or otherwise it vanishes. Assuming that A satisfies the UCT, one can then compute $KK(C_\nu A, S^{n+1}A)$ explicitly in many interesting examples. If A is a Kirchberg algebra, it is well-known that $\pi_0 U(A)\cong\pi_2 U(A)\cong K_1(A)$. In contrast, the group $\pi_0\mathrm{Aut}(A)$ needs not to be abelian and even if it is abelian, it needs not to be isomorphic to $\pi_2\mathrm{Aut}(A)$. Indeed, if A satisfies the UCT and $K_1(A)=0$, then we obtain from Corollary 5.10 that $\pi_0\mathrm{Aut}(A)$ is isomorphic to the multiplicative group $KK(A,A)_u^{-1}\cong\mathrm{Aut}(K_0(A),[1_A])$ whereas $\pi_2\mathrm{Aut}(A)$ is isomorphic to the abelian group $\mathrm{Hom}(K_0(A)/\mathbb{Z}[1_A],K_0(A))$ and hence in bijection with $\mathrm{End}(K_0(A),[1_A])$. For instance if $K_0(A)\cong\mathbb{Z}\oplus\mathbb{Z}$ and $[1_A]=0$, then $\pi_0\mathrm{Aut}(A)\cong GL(2,\mathbb{Z})$ and $\pi_2\mathrm{Aut}(A)\cong\mathbb{Z}^4$. If instead $K_0(A)\cong\mathbb{Z}$ and $[1_A]=0$, then $\pi_0\mathrm{Aut}(A)\cong\mathbb{Z}/2$ and $\pi_2\mathrm{Aut}(A)\cong\mathbb{Z}$.

6. The group $[X,\mathrm{Aut}(A)]$

In the previous section we computed $[X,\mathrm{Aut}(A)^0]$ and showed that it is an abelian group whenever X is an H'-space. On the other hand, the set of homotopy classes $[X,\mathrm{Aut}(A)]$ has a natural (not necessarily abelian) group structure whether or not X is an H'-space. In the sequel we determine this group up to an extension.

Let A and B be separable C*-algebras and let X be a path connected metrizable compact space. Let us set

$$KK(A,B)_u=\{\alpha\in KK(A,B)\,:\,K_0(\alpha)[1_A]=[1_B],\}$$

$$KK(A,B)_0=\{\alpha\in KK(A,B)\,:\,K_0(\alpha)[1_A]=[0],\}$$

$$KK(A,C(X)\otimes A)_u^*=\{\alpha\in KK(A,C(X)\otimes A)_u:\alpha_x\in KK(A,A)^{-1}\ \forall x\in X\}.$$

b Let us consider the diagram with injective vertical maps where the broken arrows indicate group actions:

$$\begin{tikzcd}
K_1(C(X)\otimes A)/\nu \arrow[r, dashed] \arrow[d, equal] & {[X,\mathrm{End}(A)]} \arrow[r, "\kappa"] & KK(A,C(X)\otimes A)_u \\
K_1(C(X)\otimes A)/\nu \arrow[r, dashed] & {[X,\mathrm{End}(A)^*]} \arrow[r, "\kappa"] \arrow[u] & KK(A,C(X)\otimes A)_u^* \arrow[u] \\
K_1(C(X,x_0)\otimes A)/\nu \arrow[r, dashed] \arrow[u] & {[X,\mathrm{End}(A)^0]} \arrow[r, "\kappa"] \arrow[u] & \iota_A+KK(A,C(X,x_0)\otimes A)_0 \arrow[u]
\end{tikzcd}$$

Proposition 6.1. *The diagram above is well-defined. The groups appearing on the left column act freely on the corresponding homotopy classes appearing on the second column. The maps κ are equivariant and surjective and their fibers coincide with the orbits of the corresponding group actions.*

Proof. The stated properties hold for the first row of the diagram by Theorem 3.6 and Remark 3.7. By the homotopy invariance of the KK-functor we see that the map $\pi_0(\mathrm{End}(A)^*) \to \pi_0(\mathrm{End}(A))$ is injective and so the vertical maps in the middle of the diagram are injective. Using the exact sequence $0 \to C(X, x_0) \to C(X) \to \mathbb{C} \to 0$ one verifies that all the other vertical maps are also injective and that one has a short exact sequence

$$0 \to K_1(C(X, x_0) \otimes A)/\nu \to K_1(C(X) \otimes A)/\nu \to K_1(A)/\nu \to 0. \tag{6}$$

In particular this establishes the claimed properties of the middle row of the diagram as a consequence of the corresponding properties of the top row. It remains to deal with the bottom row. The assumption that X is path connected is needed only for this part of the statement. Let us verify first that the map κ is well defined. Let $\varphi : X \to \mathrm{End}(A)^0$ be a continuous map. Then $\big(KK(\varphi) - \iota_A\big)_{x_0} = KK(\mathrm{id}_A) - KK(\mathrm{id}_A) = 0$ and $\varphi(1_A) - j_A(1_A) = 0$ and hence $\kappa(\varphi) \in \iota_A + KK(A, C(X, x_0) \otimes A)_0$. To verify that κ is surjective, let $\sigma \in \iota_A + KK(A, C(X, x_0) \otimes A)_0$. By the surjectivity of the map κ in the top row, there is a continuous map $\varphi : X \to \mathrm{End}(A)$ such that $KK(\varphi) = \sigma$. In particular $KK(\varphi_{x_0}) = KK(\mathrm{id}_A)$. By Theorem 3.6 and Remark 3.7 applied for $B = A$, there is a unitary $v \in U(A)$ such that $v\varphi_{x_0}v^*$ is homotopic to id_A. Since X is path connected this implies that $(v\varphi v^*)_x \in \mathrm{End}(A)^0$ for all $x \in X$ and hence $v\varphi v^*$ is a lifting of σ. Next we show $K_1(C(X, x_0) \otimes A)$ acts transitively on the fibers of κ. Let $\varphi, \psi : X \to \mathrm{End}(A)^0$ be continuous maps such that $\kappa[\varphi] = \kappa[\psi]$. By Theorem 3.6 applied for $B = C(X) \otimes A$, there is a unitary $v \in U(C(X) \otimes B)$ such that ψ is homotopic to $v\varphi v^*$. In particular ψ_{x_0} is homotopic to $v(x_0)\varphi_{x_0}\, v(x_0)^*$ and hence id_A is homotopic to $v(x_0)\mathrm{id}_A\, v(x_0)^*$. It follows by Theorem 3.6 that $[v(x_0)] = 0$ in $K_1(A)/\nu$. The exact sequence (6) shows now that the class of $[v]$ in $K_1(C(X) \otimes A)/\nu$ belongs to the image of $K_1(C(X, x_0) \otimes A)/\nu$ and hence ψ is homotopic to $w\varphi w^*$ for some $w \in U(C(X) \otimes A)$ with $w_{x_0} \in U(A)_0$. In the last part of the proof we show that $K_1(C(X, x_0) \otimes A)/\nu$ acts freely on $[X, \mathrm{End}(A)^0]$. Indeed, suppose that $w\varphi w^*$ is homotopic to φ as maps $X \to \mathrm{End}(A)^0$ where w is as above. Therefore $[w\, \varphi\, w^*] = [\varphi]$ in $[A, C(X) \otimes A]_u$ and hence $[w] = 0$ in $K_1(C(X) \otimes A)/\nu$ by Theorem 3.6. Using the exact sequence (6) again we conclude that $[w] = 0$ in $K_1(C(X, x_0) \otimes A)/\nu$. $\square$

The exact sequence of rings (5) gives rise to a split exact sequence of multiplicative groups

$$1 \longrightarrow \iota_A + K'_A(X) \longrightarrow KK(A, C(X) \otimes A)^* \xrightarrow{\rho} \check{H}^0(X, KK(A, A)^{-1}) \longrightarrow 1$$

where $\check{H}^0(X, KK(A,A)^{-1})$ denotes the locally constant maps from X to $KK(A,A)^{-1}$. If X is path connected, then $K'_A(X) \cong K_A(X, x_0) = KK(A, C(X, x_0) \otimes A)$ for every $x_0 \in X$ and $\check{H}^0(X, KK(A,A)^{-1}) = KK(A,A)^{-1}$.

Theorem 6.2. *Let A be a Kirchberg algebra and let X be a compact metrizable space such that the pair (A, X) is KK-continuous. Then there are isomorphisms of groups*

$$\kappa : [X, \mathrm{Aut}(A \otimes \mathcal{K})] \to KK(A, C(X) \otimes A)^*.$$

$$\kappa : [X, \mathrm{Aut}(A \otimes \mathcal{K})^0] \to \iota_A + K'_A(X).$$

Proof. The result follows from Propositions 2.10 and 5.8. □

Theorem 6.3. *Let A be a Kirchberg algebra and let X be a compact metrizable space such that the pair (A, X) is KK-continuous. Then there is an exact sequence of groups*

$$1 \longrightarrow K_1(C(X) \otimes A)/\nu \longrightarrow [X, \mathrm{Aut}(A)] \xrightarrow{\kappa} KK(A, C(X) \otimes A)^*_u \longrightarrow 1.$$

If X is path connected and $x_0 \in X$, then there is another exact sequence of groups:

$$1 \longrightarrow K_1(C(X, x_0) \otimes A)/\nu \longrightarrow [X, \mathrm{Aut}(A)^0] \xrightarrow{\kappa} \iota_A + KK(A, C(X, x_0) \otimes A)_0 \longrightarrow 1.$$

Proof. The result follows from Propositions 5.8 and 6.1, since the inner automorphisms of $C(X) \otimes A$ form a normal subgroup of $Aut_{C(X)}(C(X) \otimes A)$. □

Remark 6.4. If (X, x_0) is a path connected H'-space and if the pair (A, X) is KK-continuous, then $\sigma\tau = 0$ for all $\sigma, \tau \in KK(A, C(X, x_0) \otimes A)$. Indeed, by Theorems 5.9 and 6.2, $\chi(\alpha+\beta) = \chi(\alpha)+\chi(\beta)$ and $\kappa(\alpha+\beta) = \kappa(\alpha)\kappa(\beta)$. Since $\kappa = \iota_A + \chi$ we must have $\chi(\alpha)\chi(\beta) = 0$ and hence $\sigma\tau = 0$ for all $\sigma, \tau \in KK(A, C(X, x_0) \otimes A)$ since χ is bijective.

Remark 6.5. As a consequence of [9, Thm. 9.2] (whose proof relies on unpublished work of Kirchberg, see [13]), for any separable nuclear C*-algebra and any finite dimensional metric space X, an element σ is invertible in the ring $K_A(X)$ if and only if $\sigma_x \in KK(A,A)^{-1}$ for all $x \in X$. Therefore, in the statements of our main results, one can replace the assumption that the pair (X, A) is continuous by the assumption that X is finite dimensional.

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