CONTINUOUS FIELDS OF C*-ALGEBRAS OVER FINITE DIMENSIONAL SPACES

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Abstract. Let X be a finite dimensional compact metrizable space. We study a technique which employs semiprojectivity as a tool to produce approximations of $C(X)$-algebras by $C(X)$-subalgebras with controlled complexity. The following applications are given. All unital separable continuous fields of C*-algebras over X with fibers isomorphic to a fixed Cuntz algebra $\mathcal{O}_n$, $n \in \{2, 3, ..., \infty\}$ are locally trivial. They are trivial if $n = 2$ or $n = \infty$. For $n \geq 3$ finite, such a field is trivial if and only if $(n-1)[1_A] = 0$ in $K_0(A)$, where A is the C*-algebra of continuous sections of the field. We give a complete list of the Kirchberg algebras D satisfying the UCT and having finitely generated K-theory groups for which every unital separable continuous field over X with fibers isomorphic to D is automatically locally trivial or trivial. In a more general context, we show that a separable unital continuous field over X with fibers isomorphic to a KK-semiprojective Kirchberg C*-algebra is trivial if and only if it satisfies a K-theoretical Fell type condition.

Contents

1. Introduction

Gelfand's characterization of commutative C*-algebras has suggested the problem of representing non-commutative C*-algebras as sections of bundles. By a result of Fell [15], if the primitive spectrum X of a separable C*-algebra A is Hausdorff, then A is isomorphic to the C*-algebra of continuous sections vanishing at infinity of a continuous field of simple C*-algebras over X. In particular A is a continuous $C(X)$-algebra in the sense of Kasparov [18]. This description is very

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satisfactory, since as explained in [4], the continuous fields of C*-algebras are in natural correspondence with the bundles of C*-algebras in the sense of topology. Nevertheless, only a tiny fraction of the continuous fields of C*-algebras correspond to locally trivial bundles.

In this paper we prove automatic and conditional local/global trivialization results for continuous fields of Kirchberg algebras. By a Kirchberg algebra we mean a purely infinite simple nuclear separable C*-algebra [29]. Notable examples include the simple Cuntz-Krieger algebras [8]. The following theorem illustrates our results.

Theorem 1.1. *A separable unital $C(X)$-algebra A over a finite dimensional compact Hausdorff space X all of whose fibers are isomorphic to the same Cuntz algebra $\mathcal{O}_n$, $n \in \{2, 3, \dots, \infty\}$, is locally trivial. If $n = 2$ or $n = \infty$, then $A \cong C(X) \otimes \mathcal{O}_n$. If $3 \leq n < \infty$, then A is isomorphic to $C(X) \otimes \mathcal{O}_n$ if and only if $(n-1)[1_A] = 0$ in $K_0(A)$.*

The case $X = [0, 1]$ of Theorem 1.1 was proved in a joint paper with G. Elliott [10].

We parametrize the homotopy classes

$$[X, \mathrm{Aut}(\mathcal{O}_n)] \cong \begin{cases} K_1(C(X) \otimes \mathcal{O}_n) & \text{if } 3 \leq n < \infty, \\ \{*\} & \text{if } n = 2,\ \infty, \end{cases}$$

(see Theorem 7.4) and hence classify the unital separable $C(SX)$-algebras A with fiber $\mathcal{O}_n$ over the suspension SX of a finite dimensional metrizable Hausdorff space X.

To put our results in perspective, let us recall that none of the general basic properties of a continuous field implies any kind of local triviality. An example of a continuous field of Kirchberg algebras over $[0, 1]$ which is not locally trivial at any point even though all of its fibers are mutually isomorphic is exhibited in [10, Ex. 8.4]. Examples of nonexact continuous fields with similar properties were found by S. Wassermann [37].

A separable C*-algebra D is KK-semiprojective if the functor $KK(D, -)$ is continuous, see Sec. 3. The class of KK-semiprojective C*-algebras includes the nuclear semiprojective C*-algebras and also the C*-algebras which satisfy the Universal Coefficient Theorem in KK-theory (abbreviated UCT [31]) and whose K-theory groups are finitely generated. It is very interesting that the *only obstruction* to local or global triviality for a continuous field of Kirchberg algebras is of purely K-theoretical nature.

Theorem 1.2. *Let A be a separable C*-algebra whose primitive spectrum X is compact Hausdorff and of finite dimension. Suppose that each primitive quotient $A(x)$ of A is nuclear, purely infinite and stable. Then A is isomorphic to $C(X) \otimes D$ for some KK-semiprojective stable Kirchberg algebra D if and only if there is $\sigma \in KK(D, A)$ such that $\sigma_x \in KK(D, A(x))^{-1}$ for all $x \in X$. For any such σ there is an isomorphism of $C(X)$-algebras $\Phi : C(X) \otimes D \to A$ such that $KK(\Phi|_D) = \sigma$.*

We have an entirely similar result covering the unital case: Theorem 7.3. The required existence of σ is a KK-theoretical analog of the classical condition of Fell that appears in the trivialization theorem of Dixmier and Douady [12] of continuous fields with fibers isomorphic to the compact operators. An important feature of our condition is that it is a priori much weaker than the condition that A is $KK_{C(X)}$-equivalent to $C(X) \otimes D$. In particular, we do not need to worry at all about the potentially hard issue of constructing elements in $KK_{C(X)}(A, C(X) \otimes D)$. To illustrate this point, let us note that it is almost trivial to verify that the local existence of σ is automatic for unital $C(X)$-algebras with fiber $\mathcal{O}_n$ and hence to derive Theorem 1.1. A C*-algebra D has the *automatic local triviality property* if any separable $C(X)$-algebra over a finite dimensional compact Hausdorff space X all of whose fibers are isomorphic to D is locally trivial. A unital C*-algebra D has the *automatic local triviality property in the unital sense* if any separable unital $C(X)$-algebra over a finite dimensional compact Hausdorff space X all of whose fibers are isomorphic to D is locally trivial. The *automatic triviality property* is defined similarly.

Theorem 1.3. *(*Automatic triviality*) A separable continuous $C(X)$-algebra over a finite dimensional compact Hausdorff space X all of whose fibers are isomorphic to $\mathcal{O}_2 \otimes \mathcal{K}$ is isomorphic to $C(X) \otimes \mathcal{O}_2 \otimes \mathcal{K}$. The C*-algebra $\mathcal{O}_2 \otimes \mathcal{K}$ is the only Kirchberg algebra satisfying the automatic local triviality property and hence the automatic triviality property.*

Theorem 1.4. *(*Automatic local triviality in the unital sense*) A unital KK-semiprojective Kirchberg algebra D has the automatic local triviality property in the unital sense if and only if all unital $*$-endomorphisms of D are KK-equivalences. In that case, if A is a separable unital $C(X)$-algebra over a finite dimensional compact Hausdorff space X all of whose fibers are isomorphic to D, then $A \cong C(X) \otimes D$ if and only if there is $\sigma \in KK(D, A)$ such that the induced homomorphism $K_0(\sigma) : K_0(D) \to K_0(A)$ maps $[1_D]$ to $[1_A]$.*

It is natural to ask if there are other unital Kirchberg algebras besides the Cuntz algebras which have the automatic local triviality property in the unital sense. Consider the following list $\mathcal{G}$ of pointed abelian groups:

(a) $\big(\{0\}, 0\big)$; (b) $(\mathbb{Z}, k)$ with $k > 0$;

(c) $\big(\mathbb{Z}/p^{e_1} \oplus \cdots \oplus \mathbb{Z}/p^{e_n}, p^{s_1} \oplus \cdots \oplus p^{s_n}\big)$ where p is a prime, $n \geq 1$, $0 \leq s_i < e_i$ for $1 \leq i \leq n$ and $0 < s_{i+1} - s_i < e_{i+1} - e_i$ for $1 \leq i < n$. If $n = 1$ the latter condition is vacuous. Note that if the integers $1 \leq e_1 \leq \cdots \leq e_n$ are given then there exists integers $s_1, ..., s_n$ satisfying the conditions above if and only if $e_{i+1} - e_i \geq 2$ for each $1 \leq i \leq n$. If that is the case one can choose $s_i = i - 1$ for $1 \leq i \leq n$.

(d) $\big(G(p_1) \oplus \cdots \oplus G(p_m), g_1 \oplus \cdots \oplus g_m\big)$ where $p_1, ..., p_m$ are distinct primes and each $(G(p_j), g_j)$ is a pointed group as in (c).

(e) $\big(\mathbb{Z} \oplus G(p_1) \oplus \cdots \oplus G(p_m), k \oplus g_1 \oplus \cdots \oplus g_m\big)$ where $(G(p_j), g_j)$ are as in (d). Moreover we require that $k > 0$ is divisible by $p_1^{s_{n(1)}+1} \cdots p_m^{s_{n(m)}+1}$ where $s_{n(j)}$ is defined as in (c) corresponding to the prime p_j.

Theorem 1.5. *(*Automatic local triviality in the unital sense – the UCT case*) Let D be a unital Kirchberg algebra which satisfies the UCT and has finitely generated K-theory groups. (i) D has the automatic triviality property in the unital sense if and only if D is isomorphic to either $\mathcal{O}_2$ or $\mathcal{O}_\infty$. (ii) D has the automatic local triviality property in the unital sense if and only if $K_1(D) = 0$ and $(K_0(D), [1_D])$ is isomorphic to one of the pointed groups from the list $\mathcal{G}$. (iii) If D is as in (ii), then a separable unital $C(X)$-algebra A over a finite dimensional compact Hausdorff space X all of whose fibers are isomorphic to D is trivial if and only if there exists a homomorphism of groups $K_0(D) \to K_0(A)$ which maps $[1_D]$ to $[1_A]$.*

We use semiprojectivity (in various flavors) to approximate and represent continuous $C(X)$-algebras as inductive limits of fibered products of n locally trivial $C(X)$-subalgebras where $n \leq \dim(X) < \infty$. This clarifies the local structure of many $C(X)$-algebras (see Theorem 5.2) and gives a new understanding of the K-theory of separable continuous $C(X)$-algebras with arbitrary nuclear fibers.

A remarkable isomorphism result for separable nuclear strongly purely infinite stable C*-algebras was announced (with an outline of the proof) by Kirchberg in [20]: two such C*-algebras A and B with the same primitive spectrum X are isomorphic if and only if they are $KK_{C(X)}$-equivalent. This is always the case after tensoring with $\mathcal{O}_2$. However the problem of recognizing when A and B are $KK_{C(X)}$-equivalent is open even for very simple spaces X such as the unit interval or non-Hausdorff spaces with more than two points.

The proof of Theorem 4.6 (one of our main results) generalizes and refines a technique that was pioneered for fields over zero dimensional spaces in joint work with Pasnicu [11] and for fields over an interval in joint work with G. Elliott [10]. We shall rely heavily on the classification theorem (and related results) of Kirchberg [19] and Phillips [28], and on the work on non-simple nuclear purely infinite C*-algebras of Blanchard and Kirchberg [5], [4] and Kirchberg and Rørdam [21], [22].

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2. $C(X)$-algebras

Let X be a locally compact Hausdorff space. A $C(X)$-algebra is a C*-algebra A endowed with a $*$-homomorphism θ from $C_0(X)$ to the center $ZM(A)$ of the multiplier algebra $M(A)$ of A such that $C_0(X)A$ is dense in A; see [18], [3]. We write fa rather than $\theta(f)a$ for $f \in C_0(X)$ and $a \in A$. If $Y \subseteq X$ is a closed set, we let $C_0(X, Y)$ denote the ideal of $C_0(X)$ consisting of functions vanishing on Y. Then $C_0(X, Y)A$ is a closed two-sided ideal of A (by Cohen factorization). The quotient of A by this ideal is a $C(X)$-algebra denoted by $A(Y)$ and is called the restriction of $A = A(X)$ to Y. The quotient map is denoted by $\pi_Y : A(X) \to A(Y)$. If Z is a closed subset of Y we have a natural restriction map $\pi_Z^Y : A(Y) \to A(Z)$ and $\pi_Z = \pi_Z^Y \circ \pi_Y$. If Y reduces to a point x, we write $A(x)$ for $A(\{x\})$ and π_x for $\pi_{\{x\}}$. The C*-algebra $A(x)$ is called the fiber of A at x. The image $\pi_x(a) \in A(x)$ of $a \in A$ is denoted by $a(x)$. A morphism of $C(X)$-algebras $\eta : A \to B$ induces a morphism $\eta_Y : A(Y) \to B(Y)$. If $A(x) \neq 0$ for x in a dense subset of X, then θ is injective. If X is compact, then $\theta(1) = 1_{M(A)}$. Let A be a C*-algebra, $a \in A$ and $\mathcal{F}, \mathcal{G} \subseteq A$. Throughout the paper we will assume that X is a compact Hausdorff space unless stated otherwise. If $\varepsilon > 0$, we write $a \in_\varepsilon \mathcal{F}$ if there is $b \in \mathcal{F}$ such that $\|a - b\| < \varepsilon$. Similarly, we write $\mathcal{F} \subset_\varepsilon \mathcal{G}$ if $a \in_\varepsilon \mathcal{G}$ for every $a \in \mathcal{F}$. The following lemma collects some basic properties of $C(X)$-algebras.

Lemma 2.1. *Let A be a $C(X)$-algebra and let $B \subset A$ be a $C(X)$-subalgebra. Let $a \in A$ and let Y be a closed subset of X.*

- (i) *The map $x \mapsto \|a(x)\|$ is upper semi-continuous.*
- (ii) $\|\pi_Y(a)\| = \max\{\|\pi_x(a)\| : x \in Y\}$
- (iii) *If $a(x) \in \pi_x(B)$ for all $x \in X$, then $a \in B$.*
- (iv) *If $\delta > 0$ and $a(x) \in_\delta \pi_x(B)$ for all $x \in X$, then $a \in_\delta B$.*
- (v) *The restriction of $\pi_x : A \to A(x)$ to B induces an isomorphism $B(x) \cong \pi_x(B)$ for all $x \in X$.*

Proof. (i), (ii) are proved in [3] and (iii) follows from (iv). (iv): By assumption, for each $x \in X$, there is $b_x \in B$ such that $\|\pi_x(a-b_x)\| < \delta$. Using (i) and (ii), we find a closed neighborhood U_x of x such that $\|\pi_{U_x}(a-b_x)\| < \delta$. Since X is compact, there is a finite subcover (U_{x_i}). Let (α_i) be a partition of unity subordinated to this cover. Setting $b = \sum_i \alpha_i\, b_{x_i} \in B$, one checks immediately that $\|\pi_x(a-b)\| \leq \sum_i \alpha_i(x)\|\pi_x(a-b_{x_i})\| < \delta$, for all $x \in X$. Thus $\|a-b\| < \delta$ by (ii). (v): If $\iota : B \hookrightarrow A$ is the inclusion map, then $\pi_x(B)$ coincides with the image of $\iota_x : B/C(X,x)B \to A/C(X,x)A$. Thus it suffices to check that ι_x is injective. If $\iota_x(b + C(X,x)B) = \pi_x(b) = 0$ for some $b \in B$, then $b = fa$ for some $f \in C(X,x)$ and some $a \in A$. If (f_λ) is an approximate unit of $C(X,x)$, then $b = \lim_\lambda f_\lambda f a = \lim_\lambda f_\lambda b$ and hence $b \in C(X,x)B$. □

A $C(X)$-algebra such that the map $x \mapsto \|a(x)\|$ is continuous for all $a \in A$ is called a *continuous* $C(X)$-algebra or a C*-bundle [3], [23], [4]. A C*-algebra A is a continuous $C(X)$-algebra if and only if A is the C*-algebra of continuous sections of a continuous field of C*-algebras over X in the sense of [12, Def. 10.3.1], (see [3], [4], [27]).

Lemma 2.2. *Let A be a separable continuous $C(X)$-algebra over a locally compact Hausdorff space X. If all the fibers of A are nonzero, then X has a countable basis of open sets. Thus the compact subspaces of X are metrizable.*

Proof. Since A is separable, its primitive spectrum $\mathrm{Prim}(A)$ has a countable basis of open sets by [12, 3.3.4]. The continuous map $\eta : \mathrm{Prim}(A) \to X$ (induced by $\theta : C_0(X) \to ZM(A) \cong C_b(\mathrm{Prim}(A)))$ is open since the $C(X)$-algebra A is continuous and surjective since $A(x) \neq 0$ for all $x \in X$ (see [4, p. 388] and [27, Prop. 2.1, Thm. 2.3]). □

Lemma 2.3. *Let X be a compact metrizable space. A $C(X)$-algebra A all of whose fibers are nonzero and simple is continuous if and only if there is $e \in A$ such that $\|e(x)\| \geq 1$ for all $x \in X$.*

Proof. By Lemma 2.1(i) it suffices to prove that $\liminf_{n\to\infty} \|a(x_n)\| \geq \|a(x_0)\|$ for any $a \in A$ and any sequence (x_n) converging to x_0 in X. Set $D = A(x_0)$ and let e be as in the statement. Let $\psi : D \to A$ be a set-theoretical lifting of id_D such that $\|\psi(d)\| = \|d\|$ for all $d \in D$. Then $\lim_{n\to\infty} \|\pi_{x_n}\psi(a(x_0)) - a(x_n)\| = 0$ for all $a \in A$, by Lemma 2.1(i). By applying this to e, since $\|e(x_n)\| \geq 1$, we see that $\liminf_{n\to\infty} \|\pi_{x_n}\psi(e(x_0))\| \geq 1$. Since D is a simple C*-algebra, if $\varphi_n : D \to B_n$ is a sequence of contractive maps such that $\lim_{n\to\infty} \|\varphi_n(\lambda c+d) - \lambda\varphi_n(c) - \varphi_n(d)\| = 0$, $\lim_{n\to\infty} \|\varphi_n(cd) - \varphi_n(c)\varphi_n(d)\| = 0$, $\lim_{n\to\infty} \|\varphi_n(c^*) - \varphi_n(c)^*\| = 0$, for all $c, d \in D$, $\lambda \in \mathbb{C}$, and $\liminf_{n\to\infty} \|\varphi_n(c)\| > 0$ for some $c \in D$, then $\lim_{n\to\infty} \|\varphi_n(c)\| = \|c\|$ for all $c \in D$. In particular this observation applies to $\varphi_n = \pi_{x_n}\psi$ by Lemma 2.1(i). Therefore

$$\liminf_{n\to\infty} \|a(x_n)\| \geq \liminf_{n\to\infty} \left(\|\pi_{x_n}\psi(a(x_0))\| - \|\pi_{x_n}\psi(a(x_0)) - a(x_n)\|\right) = \|a(x_0)\|.$$

Conversely, if A is continuous, take e to be a large multiple of some full element of A. □

Let $\eta : B \to A$ and $\psi : E \to A$ be $*$-homomorphisms. The *pullback* of these maps is

$$B \oplus_{\eta,\psi} E = \{(b,e) \in B \oplus E : \eta(b) = \psi(e)\}.$$

We are going to use pullbacks in the context of $C(X)$-algebras. Let X be a compact space and let Y, Z be closed subsets of X such that $X = Y \cup Z$. The following result is proved in [12, Prop. 10.1.13] for continuous $C(X)$-algebras.

Lemma 2.4. *If A is a $C(X)$-algebra, then A is isomorphic to $A(Y) \oplus_{\pi,\pi} A(Z)$, the pullback of the restriction maps $\pi^Y_{Y\cap Z} : A(Y) \to A(Y \cap Z)$ and $\pi^Z_{Y\cap Z} : A(Z) \to A(Y \cap Z)$.*

Proof. By the universal property of pullbacks, the maps π_Y and π_Z induce a map $\eta : A \to A(Y) \oplus_{\pi,\pi} A(Z)$, $\eta(a) = (\pi_Y(a), \pi_Z(a))$, which is injective by Lemma 2.1(ii). Thus it suffices to show that the range of η is dense. Let $b, c \in A$ be such that $\pi_{Y\cap Z}(b - c) = 0$ and let $\varepsilon > 0$. We shall find $a \in A$ such that $\|\eta(a) - (\pi_Y(b), \pi_Z(c))\| < \varepsilon$. By Lemma 2.1(i), there is an open neighborhood V of $Y \cap Z$ such that $\|\pi_x(b - c)\| < \varepsilon$ for all $x \in V$. Let $\{\lambda, \mu\}$ be a partition of unity on X subordinated to the open cover $\{Y \cup V, Z \cup V\}$. Then $a = \lambda b + \mu c$ is an element of A which has the desired property. □

Let $B \subset A(Y)$ and $E \subset A(Z)$ be $C(X)$-subalgebras such that $\pi^Z_{Y\cap Z}(E) \subseteq \pi^Y_{Y\cap Z}(B)$. As an immediate consequence of Lemma 2.4 we see that the pullback $B \oplus_{\pi^Z_{Y\cap Z},\pi^Y_{Y\cap Z}} E$ is isomorphic to the $C(X)$-subalgebra $B \oplus_{Y\cap Z} E$ of A defined as

$$B \oplus_{Y\cap Z} E = \{a \in A : \pi_Y(a) \in B, \pi_Z(a) \in E\}.$$

Lemma 2.5. *The fibers of $B \oplus_{Y\cap Z} E$ are given by*

$$\pi_x(B \oplus_{Y\cap Z} E) = \begin{cases} \pi_x(B), & \text{if } x \in X \setminus Z, \\ \pi_x(E), & \text{if } x \in Z, \end{cases}$$

and there is an exact sequence of C-algebras*

$$0 \longrightarrow \{b \in B : \pi_{Y\cap Z}(b) = 0\} \longrightarrow B \oplus_{Y\cap Z} E \xrightarrow{\pi_Z} E \longrightarrow 0 \tag{1}$$

Proof. Let $x \in X \setminus Z$. The inclusion $\pi_x(B \oplus_{Y\cap Z} E) \subset \pi_x(B)$ is obvious by definition. Given $b \in B$, let us choose $f \in C(X)$ vanishing on Z and such that $f(x) = 1$. Then $a = (fb, 0)$ is an element of A by Lemma 2.4. Moreover $a \in B \oplus_{Y\cap Z} E$ and $\pi_x(a) = \pi_x(b)$. We have $\pi_Z(B \oplus_{Y\cap Z} E) \subset E$, by definition. Conversely, given $e \in E$, let us observe that $\pi^Z_{Y\cap Z}(e) \in \pi^Y_{Y\cap Z}(B)$ (by assumption) and hence $\pi^Z_{Y\cap Z}(e) = \pi^Y_{Y\cap Z}(b)$ for some $b \in B$. Then $a = (b, e)$ is an element of A by Lemma 2.4 and $\pi_Z(a) = e$. This completes the proof for the first part of the lemma and also it shows that the map π_Z from the sequence (1) is surjective. Its kernel is identified using Lemma 2.1(iii). □

Let X, Y, Z and A be as above. Let $\eta : B \hookrightarrow A(Y)$ be a $C(Y)$-linear $*$-monomorphism and let $\psi : E \hookrightarrow A(Z)$ be a $C(Z)$-linear $*$-monomorphism. Assume that

$$\pi^Z_{Y\cap Z}(\psi(E)) \subseteq \pi^Y_{Y\cap Z}(\eta(B)). \tag{2}$$

This gives a map $\gamma = \eta^{-1}_{Y\cap Z}\psi_{Y\cap Z} : E(Y \cap Z) \to B(Y \cap Z)$. To simplify notation we let π stand for both $\pi^Y_{Y\cap Z}$ and $\pi^Z_{Y\cap Z}$ in the following lemma.

Lemma 2.6. *(a) There are isomorphisms of $C(X)$-algebras*

$$B \oplus_{\pi,\gamma\pi} E \cong B \oplus_{\pi\eta,\pi\psi} E \cong \eta(B) \oplus_{Y\cap Z} \psi(E),$$

where the second isomorphism is given by the map $\chi : B \oplus_{\pi\eta,\pi\psi} E \to A$ induced by the pair (η, ψ). Its components χ_x can be identified with ψ_x for $x \in Z$ and with η_x for $x \in X \setminus Z$.

(b) Condition (2) is equivalent to $\psi(E) \subset \pi_Z\big(A \oplus_Y \eta(B)\big)$.

(c) If $\mathcal{F}$ is a finite subset of A such that $\pi_Y(\mathcal{F}) \subset_\varepsilon \eta(B)$ and $\pi_Z(\mathcal{F}) \subset_\varepsilon \psi(E)$, then $\mathcal{F} \subset_\varepsilon \eta(B) \oplus_{Y\cap Z} \psi(E) = \chi\big(B \oplus_{\pi\eta,\pi\psi} E\big)$.

Proof. This is an immediate corollary of Lemmas 2.1, 2.4, 2.5. For illustration, let us verify (c). By assumption $\pi_x(\mathcal{F}) \subset_\varepsilon \eta_x(B)$ for all $x \in X \setminus Z$ and $\pi_z(\mathcal{F}) \subset_\varepsilon \psi_z(E)$ for all $z \in Z$. We deduce from Lemma 2.5 that $\pi_x(\mathcal{F}) \subset_\varepsilon \pi_x(\eta(B) \oplus_{Y\cap Z} \psi(E))$ for all $x \in X$. Therefore $\mathcal{F} \subset_\varepsilon \eta(B) \oplus_{Y\cap Z} \psi(E)$ by Lemma 2.1(iv). □

Definition 2.7. Let $\mathcal{C}$ be a class of C*-algebras. A $C(Z)$-algebra E is called $\mathcal{C}$-*elementary* if there is a finite partition of Z into closed subsets $Z_1, \dots, Z_r$ $(r \geq 1)$ and there exist C*-algebras $D_1, \dots, D_r$ in $\mathcal{C}$ such that $E \cong \bigoplus_{i=1}^r C(Z_i) \otimes D_i$. The notion of *category* of a $C(X)$-algebra with respect to a class $\mathcal{C}$ is defined inductively: if A is $\mathcal{C}$-elementary then $\mathrm{cat}_{\mathcal{C}}(A) = 0$; $\mathrm{cat}_{\mathcal{C}}(A) \leq n$ if there are closed subsets Y and Z of X with $X = Y \cup Z$ and there exist a $C(Y)$-algebra B such that $\mathrm{cat}_{\mathcal{C}}(B) \leq n-1$, a $\mathcal{C}$-elementary $C(Z)$-algebra E and a $*$-monomorphism of $C(Y\cap Z)$-algebras $\gamma : E(Y\cap Z) \to B(Y\cap Z)$ such that A is isomorphic to

$$B \oplus_{\pi,\gamma\pi} E = \{(b,d) \in B \oplus E : \pi^Y_{Y\cap Z}(b) = \gamma\pi^Z_{Y\cap Z}(d)\}.$$

By definition $\mathrm{cat}_{\mathcal{C}}(A) = n$ if n is the smallest number with the property that $\mathrm{cat}_{\mathcal{C}}(A) \leq n$. If no such n exists, then $\mathrm{cat}_{\mathcal{C}}(A) = \infty$.

Definition 2.8. Let $\mathcal{C}$ be a class of C*-algebras and let A be a $C(X)$-algebra. An *n-fibered $\mathcal{C}$-monomorphism* $(\psi_0, \dots, \psi_n)$ *into* A consists of $(n+1)$ $*$-monomorphisms of $C(X)$-algebras $\psi_i : E_i \to A(Y_i)$, where $Y_0, \dots, Y_n$ is a closed cover of X, each E_i is a $\mathcal{C}$-elementary $C(Y_i)$-algebra and

$$\pi^{Y_i}_{Y_i\cap Y_j}\psi_i(E_i) \subseteq \pi^{Y_j}_{Y_i\cap Y_j}\psi_j(E_j), \qquad \text{for all } i \leq j. \tag{3}$$

Given an n-fibered morphism into A we have an associated *continuous* $C(X)$-algebra defined as the fibered product (or pullback) of the $*$-monomorphisms ψ_i:

$$A(\psi_0, \dots, \psi_n) = \{(d_0, \dots d_n) : d_i \in E_i,\ \pi^{Y_i}_{Y_i\cap Y_j}\psi_i(d_i) = \pi^{Y_j}_{Y_i\cap Y_j}\psi_j(d_j) \text{ for all } i,j\} \tag{4}$$

and an induced $C(X)$-monomorphism (defined by using Lemma 2.4)

$$\eta = \eta_{(\psi_0,\dots,\psi_n)} : A(\psi_0, \dots, \psi_n) \to A \subset \bigoplus_{i=0}^n A(Y_i),$$

$$\eta(d_0, \dots d_n) = \big(\psi_0(d_0), \dots, \psi_n(d_n)\big).$$

There are natural coordinate maps $p_i : A(\psi_0, \dots, \psi_n) \to E_i$, $p_i(d_0, \dots, d_n) = d_i$. Let us set $X_k = Y_k \cup \dots \cup Y_n$. Then $(\psi_k, \dots \psi_n)$ is an $(n-k)$-fibered $\mathcal{C}$-monomorphism into $A(X_k)$. Let $\eta_k : A(X_k)(\psi_k, \dots \psi_n) \to A(X_k)$ be the induced map and set $B_k = A(X_k)(\psi_k, \dots \psi_n)$. Let us note that $B_0 = A(\psi_0, \dots, \psi_n)$ and that there are natural $C(X_{k-1})$-isomorphisms

$$B_{k-1} \cong B_k \oplus_{\pi\eta_k, \pi\psi_{k-1}} E_{k-1} \cong B_k \oplus_{\pi,\gamma_k\pi} E_{k-1}. \tag{5}$$

where π stands for $\pi_{X_k\cap Y_{k-1}}$ and $\gamma_k : E_{k-1}(X_k \cap Y_{k-1}) \to B_k(X_k \cap Y_{k-1})$ is defined by $(\gamma_k)_x = (\eta_k)^{-1}_x(\psi_{k-1})_x$, for all $x \in X_k \cap Y_{k-1}$. In particular, this decomposition shows that $\mathrm{cat}_{\mathcal{C}}(A(\psi_0, \dots, \psi_n)) \leq n$.

Lemma 2.9. *Suppose that the class $\mathcal{C}$ from Definition 2.7 consists of stable Kirchberg algebras. If A is a $C(X)$-algebra over a compact metrizable space X such that $\mathrm{cat}_{\mathcal{C}}(A) < \infty$, then A contains a full properly infinite projection and $A \cong A \otimes \mathcal{O}_\infty \otimes \mathcal{K}$.*

Proof. We prove this by induction on $n = \mathrm{cat}_{\mathcal{C}}(A)$. The case $n = 0$ is immediate since $D \cong D \otimes \mathcal{O}_\infty$ for any Kirchberg algebra D [19]. Let $A = B \oplus_{\pi,\gamma\pi} E$ where B, E and γ are as in Definition 2.7 with $\mathrm{cat}_{\mathcal{C}}(B) = n-1$ and $\mathrm{cat}_{\mathcal{C}}(E) = 0$. Let us consider the exact sequence $0 \to J \to A \to E \to 0$, where $J = \{b \in B : \pi_{Y\cap Z}(b) = 0\}$. Since J is an ideal of $B \cong B \otimes \mathcal{O}_\infty \otimes \mathcal{K}$, J absorbs $\mathcal{O}_\infty \otimes \mathcal{K}$ by [22, Prop. 8.5]. Since both E and J are stable and purely infinite, it follows that A is stable by [30, Prop. 6.12] and purely infinite by [22, Prop. 3.5]. Since A has Hausdorff primitive spectrum, A is strongly purely infinite by [5, Thm. 5.8]. It follows that $A \cong A \otimes \mathcal{O}_\infty$ by [22, Thm. 9.1]. Finally A contains a full properly infinite projection since there is a full embedding of $\mathcal{O}_2$ into A by [5, Prop. 5.6]. □

3. Semiprojectivity

In this section we study the notion of KK-semiprojectivity. The main result is Theorem 3.12. Let A and B be C*-algebras. Two $*$-homomorphisms $\varphi, \psi : A \to B$ are approximately unitarily equivalent, written $\varphi \approx_u \psi$, if there is a sequence of unitaries (u_n) in the C*-algebra $B^+ = B + \mathbb{C}1$ obtained by adjoining a unit to B, such that $\lim_{n\to\infty} \|u_n\varphi(a)u_n^* - \psi(a)\| = 0$ for all $a \in A$. We say that φ and ψ are asymptotically unitarily equivalent, written $\varphi \approx_{uh} \psi$, if there is a norm continuous unitary valued map $t \to u_t \in B^+$, $t \in [0,1)$, such that $\lim_{t\to 1} \|u_t\varphi(a)u_t^* - \psi(a)\| = 0$ for all $a \in A$. A $*$-homomorphism $\varphi : D \to A$ is full if $\varphi(d)$ is not contained in any proper two-sided closed ideal of A if $d \in D$ is nonzero.

We shall use several times Kirchberg's Theorem [29, Thm. 8.3.3] and the following theorem of Phillips [28].

Theorem 3.1. *Let A and B be separable C*-algebras such that A is simple and nuclear, $B \cong B \otimes \mathcal{O}_\infty$, and there exist full projections $p \in A$ and $q \in B$. For any $\sigma \in KK(A,B)$ there is a full $*$-homomorphism $\varphi : A \to B$ such that $KK(\varphi) = \sigma$. If $K_0(\sigma)[p] = [q]$ then we may arrange that $\varphi(p) = q$. If $\psi : A \to B$ is another $*$-homomorphism such that $KK(\psi) = KK(\varphi)$ and $\psi(p) = q$, then $\varphi \approx_{uh} \psi$ via a path of unitaries $t \mapsto u_t \in U(qBq)$.*

Theorem 3.1 does not appear in this form in [28] but it is an immediate consequence of [28, Thm. 4.1.1]. Since $pAp \otimes \mathcal{K} \cong A \otimes \mathcal{K}$ and $qBq \otimes \mathcal{K} \cong B \otimes \mathcal{K}$ by [6], and $qBq \otimes \mathcal{O}_\infty \cong qBq$ by [22, Prop. 8.5], it suffices to discuss the case when p and q are the units of A and B. If σ is given, [28, Thm. 4.1.1] yields a full $*$-homomorphism $\varphi : A \to B \otimes \mathcal{K}$ such that $KK(\varphi) = \sigma$. Let $e \in \mathcal{K}$ be a rank-one projection and suppose that $[\varphi(1_A)] = [1_B \otimes e]$ in $K_0(B)$. Since both $\varphi(1_A)$ and $1_B \otimes e$ are full projections and $B \cong B \otimes \mathcal{O}_\infty$, it follows by [28, Lemma 2.1.8] that $u\varphi(1_A)u^* = 1_B \otimes e$ for some unitary in $(B \otimes \mathcal{K})^+$. Replacing φ by $u\,\varphi\,u^*$ we can arrange that $KK(\varphi) = \sigma$ and φ is unital. For the second part of the theorem let us note that any unital $*$-homomorphism $\varphi : A \to B$ is full and if two unital $*$-homomorphisms $\varphi, \psi : A \to B$ are asymptotically unitarily equivalent when regarded as maps into $B \otimes \mathcal{K}$, then $\varphi \approx_{uh} \psi$ when regarded as maps into B, by an argument from the proof of [28, Thm. 4.1.4].

A separable nonzero C*-algebra D is *semiprojective* [1] if for any separable C*-algebra A and any increasing sequence of two-sided closed ideals (J_n) of A with $J = \overline{\cup_n J_n}$, the natural map $\varinjlim \mathrm{Hom}(D, A/J_n) \to \mathrm{Hom}(D, A/J)$ (induced by $\pi_n : A/J_n \to A/J$) is surjective. If we weaken this condition and require only that the above map has dense range, where $\mathrm{Hom}(D, A/J)$ is given the point-norm topology, then D is called *weakly semiprojective* [14]. These definitions do not change if we drop the separability of A. We shall use (weak) semiprojectivity in the following context. Let A be a $C(X)$-algebra (with X metrizable), let $x \in X$ and set $U_n = \{y \in X : d(y,x) \leq 1/n\}$. Then $J_n = C(X, U_n)A$ is an increasing sequence of ideals of A such that $J = C(X,x)A$, $A/J_n \cong A(U_n)$ and $A/J \cong A(x)$.

Examples 3.2. (Weakly semiprojective C*-algebras) Any finite dimensional C*-algebra is semiprojective. A Kirchberg algebra D satisfying the UCT and having finitely generated K-theory groups is weakly semiprojective by work of Neubüser [26], H. Lin [24] and Spielberg [32]. This also follows from Theorem 3.12 and Proposition 3.14 below. If in addition $K_1(D)$ is torsion free, then D is semiprojective as proved by Spielberg [33] who extended the foundational work of Blackadar [1] and Szymanski [34].

The following generalizations of two results of Loring [25] are used in section 5; see [10].

Proposition 3.3. *Let D be a separable semiprojective C*-algebra. For any finite set $\mathcal{F} \subset D$ and any $\varepsilon > 0$, there exist a finite set $\mathcal{G} \subset D$ and $\delta > 0$ with the following property. Let $\pi : A \to B$ be a surjective $*$-homomorphism, and let $\varphi : D \to B$ and $\gamma : D \to A$ be $*$-homomorphisms such that $\|\pi\gamma(d) - \varphi(d)\| < \delta$ for all $d \in \mathcal{G}$. Then there is a $*$-homomorphism $\psi : D \to A$ such that $\pi\psi = \varphi$ and $\|\gamma(c) - \psi(c)\| < \varepsilon$ for all $c \in \mathcal{F}$.*

Proposition 3.4. *Let D be a separable semiprojective C*-algebra. For any finite set $\mathcal{F} \subset D$ and any $\varepsilon > 0$, there exist a finite set $\mathcal{G} \subset D$ and $\delta > 0$ with the following property. For any two $*$-homomorphisms $\varphi, \psi : D \to B$ such that $\|\varphi(d) - \psi(d)\| < \delta$ for all $d \in \mathcal{G}$, there is a homotopy $\Phi \in \mathrm{Hom}(D, C[0,1] \otimes B)$ such that $\Phi_0 = \varphi$ to $\Phi_1 = \psi$ and $\|\varphi(c) - \Phi_t(c)\| < \varepsilon$ for all $c \in \mathcal{F}$ and $t \in [0,1]$.*

Definition 3.5. A separable C*-algebra D is KK-stable if there is a finite set $\mathcal{G} \subset D$ and there is $\delta > 0$ with the property that for any two $*$-homomorphisms $\varphi, \psi : D \to A$ such that $\|\varphi(a) - \psi(a)\| < \delta$ for all $a \in \mathcal{G}$, one has $KK(\varphi) = KK(\psi)$.

Corollary 3.6. *Any semiprojective C*-algebra is weakly semiprojective and KK-stable.*

Proof. This follows from Proposition 3.4. □

Proposition 3.7. *Let D be a separable weakly semiprojective C*-algebra. For any finite set $\mathcal{F} \subset D$ and any $\varepsilon > 0$ there exist a finite set $\mathcal{G} \subset D$ and $\delta > 0$ such that for any C*-algebras $B \subset A$ and any $*$-homomorphism $\varphi : D \to A$ with $\varphi(\mathcal{G}) \subset_\delta B$, there is a $*$-homomorphism $\psi : D \to B$ such that $\|\varphi(c) - \psi(c)\| < \varepsilon$ for all $c \in \mathcal{F}$. If in addition D is KK-stable, then we can choose $\mathcal{G}$ and δ such that we also have $KK(\psi) = KK(\varphi)$.*

Proof. This follows from [14, Thms. 3.1, 4.6]. Since the result is essential to us we include a short proof. Fix $\mathcal{F}$ and ε. Let $(\mathcal{G}_n)$ be an increasing sequence of finite subsets of D whose union is dense in D. If the statement is not true, then there are sequences of C*-algebras $C_n \subset A_n$ and $*$-homomorphisms $\varphi_n : D \to A_n$ satisfying $\varphi_n(\mathcal{G}_n) \subset_{1/n} C_n$ and with the property that for any $n \geq 1$ there is no $*$-homomorphism $\psi_n : D \to C_n$ such that $\|\varphi_n(c) - \psi_n(c)\| < \varepsilon$ for all $c \in \mathcal{F}$. Set $B_i = \prod_{n \geq i} A_n$ and $E_i = \prod_{n \geq i} C_n \subset B_i$. If $\nu_i : B_i \to B_{i+1}$ is the natural projection, then $\nu_i(E_i) = E_{i+1}$. Let us observe that if we define $\Phi_i : D \to B_i$ by $\Phi_i(d) = (\varphi_i(d), \varphi_{i+1}(d), \dots)$, then the image of $\Phi = \varinjlim \Phi_i : D \to \varinjlim (B_i, \nu_i)$ is contained in $\varinjlim (E_i, \nu_i)$. Since D is weakly semiprojective, there is i and a $*$-homomorphism $\Psi_i : D \to E_i$, of the form $\Psi_i(d) = (\psi_i(d), \psi_{i+1}(d), \dots)$ such that $\|\Phi_i(c) - \Psi_i(c)\| < \varepsilon$ for all $c \in \mathcal{F}$. Therefore $\|\varphi_i(c) - \psi_i(c)\| < \varepsilon$ for all $c \in \mathcal{F}$ which gives a contradiction. □

It is useful to combine Propositions 3.7 and 3.3 in a single statement.

Proposition 3.8. *Let D be a separable semiprojective C*-algebra. For any finite set $\mathcal{F} \subset D$ and any $\varepsilon > 0$, there exist a finite set $\mathcal{G} \subset D$ and $\delta > 0$ with the following property. Let $\pi : A \to B$ be a surjective $*$-homomorphism which maps a C*-subalgebra A' of A onto a C*-subalgebra B' of B. Let $\varphi : D \to B'$ and $\gamma : D \to A$ be $*$-homomorphisms such that $\gamma(\mathcal{G}) \subset_\delta A'$ and $\|\pi\gamma(d) - \varphi(d)\| < \delta$ for all $d \in \mathcal{G}$. Then there is a $*$-homomorphism $\psi : D \to A'$ such that $\pi\psi = \varphi$ and $\|\gamma(c) - \psi(c)\| < \varepsilon$ for all $c \in \mathcal{F}$.*

Proof. Let $\mathcal{G}_L$ and δ_L be given by Proposition 3.3 applied to the input data $\mathcal{F}$ and $\varepsilon/2$. We may assume that $\mathcal{F} \subset \mathcal{G}_L$ and $\varepsilon > \delta_L$. Next, let $\mathcal{G}_P$ and δ_P be given by Proposition 3.7 applied to the input data $\mathcal{G}_L$ and $\delta_L/2$. We show now that $\mathcal{G} := \mathcal{G}_L \cup \mathcal{G}_P$ and $\delta := \min\{\delta_P, \delta_L/2\}$ have the desired properties. We have $\gamma(\mathcal{G}_P) \subset_{\delta_P} A'$ since $\mathcal{G}_P \subset \mathcal{G}$ and $\delta \le \delta_L$. By Proposition 3.7 there is a $*$-homomorphism $\gamma' : D \to A'$ such that $\|\gamma'(d) - \gamma(d)\| < \delta_L/2$ for all $d \in \mathcal{G}_L$. Then

$$\|\pi\gamma'(d) - \varphi(d)\| \le \|\pi\gamma'(d) - \pi\gamma(d)\| + \|\pi\gamma(d) - \varphi(d)\| < \delta_L/2 + \delta \le \delta_L$$

for all $d \in \mathcal{G}_L$ since $\mathcal{G}_L \subset \mathcal{G}$ and $\delta \le \delta_L/2$. Therefore we can invoke Proposition 3.3 to perturb γ' to a $*$-homomorphism $\psi : D \to A'$ such that $\pi\psi = \varphi$ and $\|\gamma'(d) - \psi(d)\| < \varepsilon/2$ for all $d \in \mathcal{F}$. Finally we observe that for $d \in \mathcal{F} \subset \mathcal{G}_L$

$$\|\gamma(d) - \psi(d)\| \le \|\gamma(d) - \gamma'(d)\| + \|\gamma'(d) - \psi(d)\| < \delta_L/2 + \varepsilon/2 < \varepsilon.$$

□

Definition 3.9. (a) A separable C*-algebra D is *KK-semiprojective* if for any separable C*-algebra A and any increasing sequence of two-sided closed ideals (J_n) of A with $J = \overline{\cup_n J_n}$, the natural map $\varinjlim KK(D, A/J_n) \to KK(D, A/J)$ is surjective.

(b) We say that the functor $KK(D, -)$ is *continuous* if for any inductive system $B_1 \to B_2 \to ...$ of separable C*-algebras, the induced map $\varinjlim KK(D, B_n) \to KK(D, \varinjlim B_n)$ is bijective.

Proposition 3.10. *Any separable KK-semiprojective C*-algebra is KK-stable.*

Proof. We shall prove the statement by contradiction. Let D be separable KK-semiprojective C*-algebra. Let $(\mathcal{G}_n)$ be an increasing sequence of finite subsets of D whose union is dense in D. If the statement is not true, then there are sequences of $*$-homomorphisms $\varphi_n, \psi_n : D \to A_n$ such that $\|\varphi_n(d) - \psi_n(d)\| < 1/n$ for all $d \in \mathcal{G}_n$ and yet $KK(\varphi_n) \neq KK(\psi_n)$ for all $n \ge 1$. Set $B_i = \prod_{n \ge i} A_n$ and let $\nu_i : B_i \to B_{i+1}$ be the natural projection. Let us define $\Phi_i, \Psi_i : D \to B_i$ by $\Phi_i(d) = (\varphi_i(d), \varphi_{i+1}(d), \dots)$ and $\Psi_i(d) = (\psi_i(d), \psi_{i+1}(d), \dots)$, for all d in D. Let B_i' be the separable C*-subalgebra of B_i generated by the images of Φ_i and Ψ_i. Then $\nu_i(B_i') = B_{i+1}'$ and one verifies immediately that $\varinjlim \Phi_i = \varinjlim \Psi_i : D \to \varinjlim (B_i', \nu_i)$. Since D is KK-semiprojective, we must have $KK(\Phi_i) = KK(\Psi_i)$ for some i and hence $KK(\varphi_n) = KK(\psi_n)$ for all $n \ge i$. This gives a contradiction. □

Proposition 3.11. *A unital Kirchberg algebra D is KK-stable if and only if $D \otimes \mathcal{K}$ is KK-stable. D is weakly semiprojective if and only if $D \otimes \mathcal{K}$ is weakly semiprojective.*

Proof. Since $KK(D,A)\cong KK(D,A\otimes\mathcal{K})\cong KK(D\otimes\mathcal{K},A\otimes\mathcal{K})$ the first part of the proposition is immediate. Suppose now that $D\otimes\mathcal{K}$ is weakly semiprojective. Then D is weakly semiprojective as shown in the proof of [32, Thm. 2.2]. Conversely, assume that D is weakly semiprojective. It suffices to find $\alpha\in\mathrm{Hom}(D\otimes\mathcal{K},D)$ and a sequence (β_n) in $\mathrm{Hom}(D,D\otimes\mathcal{K})$ such that $\beta_n\alpha$ converges to $\mathrm{id}_{D\otimes\mathcal{K}}$ in the point-norm topology. Let s_i be the canonical generators of $\mathcal{O}_\infty$. If (e_{ij}) is a system of matrix units for $\mathcal{K}$, then $\lambda(e_{ij})=s_is_j^*$ defines a $*$-homomorphism $\mathcal{K}\to\mathcal{O}_\infty$ such that $KK(\lambda)\in KK(\mathcal{K},\mathcal{O}_\infty)^{-1}$. Therefore, by composing $\mathrm{id}_D\otimes\lambda$ with some isomorphism $D\otimes\mathcal{O}_\infty\cong D$ (given by [29, Thm. 7.6.6]) we obtain a $*$-monomorphism $\alpha: D\otimes\mathcal{K}\to D$ which induces a KK-equivalence. Let $\beta: D\to D\otimes\mathcal{K}$ be defined by $\beta(d)=d\otimes e_{11}$. Then $\beta\alpha\in\mathrm{End}(D\otimes\mathcal{K})$ induces a KK-equivalence and hence after replacing β by $\theta\beta$ for some automorphism θ of $D\otimes\mathcal{K}$, we may arrange that $KK(\beta\alpha)=KK(\mathrm{id}_D)$. By Theorem 3.1, $\beta\alpha\approx_u\mathrm{id}_{D\otimes\mathcal{K}}$, so that there is a sequence of unitaries $u_n\in(D\otimes\mathcal{K})^+$ such that $u_n\beta\alpha(-)u_n^*$ converges to $\mathrm{id}_{D\otimes\mathcal{K}}$. □

Theorem 3.12. *For a separable C*-algebra D consider the following properties:*

(i) D is KK-semiprojective.

(ii) The functor $KK(D,-)$ is continuous.

(iii) D is weakly semiprojective and KK-stable.

Then (i) ⇔ (ii). Moreover, (iii) ⇒ (i) if D is nuclear and (i) ⇒ (iii) if D is a Kirchberg algebra. Thus (i) ⇔ (ii) ⇔ (iii) for any Kirchberg algebra D.

Proof. The implication (ii) ⇒ (i) is obvious. (i) ⇒ (ii): Let $(B_n,\gamma_{n,m})$ be an inductive system with inductive limit B and let $\gamma_n: B_n\to B$ be the canonical maps. We have an induced map $\beta:\varinjlim KK(D,B_n)\to KK(D,B)$. First we show that β is surjective. The mapping telescope construction of L. G. Brown (as described in the proof of [1, Thm. 3.1]) produces an inductive system of C*-algebras $(T_n,\eta_{n,m})$ with inductive limit B such that each $\eta_{n,n+1}$ is surjective, and each canonical map $\eta_n: T_n\to B$ is homotopic to $\gamma_n\alpha_n$ for some $*$-homomorphism $\alpha_n: T_n\to B_n$. In particular $KK(\eta_n)=KK(\gamma_n)KK(\alpha_n)$. Let $x\in KK(D,B)$. By (i) there are n and $y\in KK(D,T_n)$ such that $KK(\eta_n)y=x$ and hence $KK(\gamma_n)KK(\alpha_n)y=x$. Thus $z=KK(\alpha_n)y\in KK(D,B_n)$ is a lifting of x. Let us show now that the map β is injective. Let x be an element in the kernel of the map $KK(D,B_n)\to KK(D,B)$. Consider the commutative diagram whose exact rows are portions of the Puppe sequence in KK-theory [2, Thm. 19.4.3] and with vertical maps induced by $\gamma_m: B_m\to B$, $m\geq n$.

$$\begin{tikzcd}
KK(D,C_{\gamma_n}) \arrow[r] & KK(D,B_n) \arrow[r] & KK(D,B) \\
KK(D,C_{\gamma_{n,m}}) \arrow[r] \arrow[u] & KK(D,B_n) \arrow[r] \arrow[u, equal] & KK(D,B_m) \arrow[u]
\end{tikzcd}$$

By exactness, x is the image of some element $y\in KK(D,C_{\gamma_n})$. Since $C_{\gamma_n}=\varinjlim C_{\gamma_{n,m}}$, the map $\varinjlim KK(D,C_{\gamma_{n,m}})\to KK(D,C_{\gamma_n})$ is surjective by the first part of the proof. Therefore there is $m\geq n$ such that y lifts to some $z\in KK(D,C_{\gamma_{n,m}})$. The image of z in $KK(D,B_m)$ equals $KK(\gamma_{n,m})x$ and vanishes by exactness of the bottom row.

(iii) ⇒ (i): Let A, (J_n) and J be as in Definition 3.9. Using the five-lemma and the split exact sequence $0\to KK(D,A)\to KK(D,A^+)\to KK(D,\mathbb{C})\to 0$, we reduce the proof to the case when A is unital. Let $x\in KK(D,A/J)$. Since the map $KK(D^+,A/J)\to KK(D,A/J)$ is surjective, x lifts to some element $x^+ \in KK(D^+, A/J)$. By [29, Thm. 8.3.3], since D^+ is nuclear, there is a $*$-homomorphism $\Phi : D^+ \to A/J \otimes \mathcal{O}_\infty \otimes \mathcal{K}$ such that $KK(\Phi) = x^+$ and hence if set $\varphi = \Phi|_D$, then $KK(\varphi) = x$. Since D is weakly semiprojective, there are n and a $*$-homomorphism $\psi : D \to A/J_n \otimes \mathcal{O}_\infty \otimes \mathcal{K}$ such that $\|\pi_n\psi(d) - \varphi(d)\| < \delta$ for all $d \in \mathcal{G}$, where $\mathcal{G}$ and δ are as in the definition of KK-stability. Therefore $KK(\pi_n\psi) = KK(\varphi)$ and hence $KK(\psi)$ is a lifting of x to $KK(D, A/J_n)$.

(i) $\Rightarrow$ (iii): D is KK-stable by Proposition 3.10. It remains to show that D is weakly semiprojective. Since any nonunital Kirchberg algebra is isomorphic to the stabilization of a unital one (see [29, Prop. 4.1.3]) and since by Proposition 3.11 D is KK-semiprojective if and only if $D \otimes \mathcal{K}$ is KK-semiprojective, we may assume that D is unital. Let A, (J_n), $\pi_{m,n} : A/J_m \to A/J_n$ $(m \leq n)$ and $\pi_n : A/J_n \to A/J$ be as in the definition of weak semiprojectivity. By [1, Cor. 2.15], we may assume that A and the $*$-homomorphism $\varphi : D \to A$ (for which we want to construct an approximative lifting) are unital. In particular φ is injective since D is simple. Set $B = \varphi(D) \subset A/J$ and $B_n = \pi_n^{-1}(B) \subset A/J_n$. The corresponding maps $\pi_{m,n} : B_m \to B_n$ $(m \leq n)$ and $\pi_n : B_n \to B$ are surjective and they induce an isomorphism $\varinjlim (B_n, \pi_{n,n+1}) \cong B$.

Given $\varepsilon > 0$ and $\mathcal{F} \subset D$ (a finite set) we are going to produce an approximate lifting $\varphi_n : D \to B_n$ for φ. Since 1_B is a properly infinite projection, it follows by [1, Props. 2.18 and 2.23] that the unit 1_n of B_n is a properly infinite projection, for all sufficiently large n. Since D is KK-semiprojective, there exist m and an element $h \in KK(D, B_m)$ which lifts $KK(\varphi)$ such that $K_0(h)[1_D] = [1_m]$. By [29, Thm. 8.3.3], there is a full $*$-homomorphism $\eta : D \to B_m \otimes \mathcal{K}$ such that $KK(\eta) = h$. By [29, Prop. 4.1.4], since both $\eta(1_D)$ and 1_m are full and properly infinite projections in $B_m \otimes \mathcal{K}$, there is a partial isometry $w \in B_m \otimes \mathcal{K}$ such that $w^*w = \eta(1_D)$ and $ww^* = 1_m$. Replacing η by $w\,\eta(-)\,w^*$, we may assume that $\eta : D \to B_m$ is unital. Then $KK(\pi_m\eta) = KK(\pi_m)h = KK(\varphi)$. By Theorem 3.1, $\pi_m\eta \approx_{uh} \varphi$. Thus there is a unitary $u \in B$ such that $\|u\pi_m\eta(d)u^* - \varphi(d)\| < \varepsilon$ for all $d \in \mathcal{F}$. Since $C(\mathbb{T})$ is semiprojective, there is $n \geq m$ such that u lifts to a unitary unitary $u_n \in B_n$. Then $\varphi_n = u_n\,\pi_{m,n}\eta(-)\,u_n^*$ is a $*$-homomorphism from D to B_n such that $\|\pi_n\varphi_n(d) - \varphi(d)\| < \varepsilon$ for all $d \in \mathcal{F}$. □

Corollary 3.13. *Any separable nuclear semiprojective C*-algebra is KK-semiprojective.*

Proof. This is very similar to the proof of the implication (iii) $\Rightarrow$ (i) of Theorem 3.12. Alternatively, the statement follows from Corollary 3.6 and Theorem 3.12. □

Blackadar has shown that a semiprojective Kirchberg algebra satisfying the UCT has finitely generated K-theory groups [29, Prop. 8.4.15]. A similar argument gives the following:

Proposition 3.14. *Let D be a separable C*-algebra satisfying the UCT. Then D is KK-semiprojective if and only $K_*(D)$ is finitely generated.*

Proof. If $K_*(D)$ is finitely generated, then D is KK-semiprojective by [31]. Conversely, assume that D is KK-semiprojective. Since D satisfies the UCT, we infer that if $G = K_i(D)$ $(i = 0, 1)$, then G is semiprojective in the category of countable abelian groups, in the sense that if $H_1 \to H_2 \to \cdots$ is an inductive system of countable abelian groups with inductive limit H, then the natural map $\varinjlim \mathrm{Hom}(G, H_n) \to \mathrm{Hom}(G, H)$ is surjective. This implies that G is finitely generated. Indeed, taking $H = G$, we see that id_G lifts to $\mathrm{Hom}(G, H_n)$ for some finitely generated subgroup H_n of G and hence G is a quotient of H_n. □

4. Approximation of $C(X)$-algebras

In this section we use weak semiprojectivity to approximate a $C(X)$-algebra A by $C(X)$-subalgebras given by pullbacks of n-fibered monomorphisms into A.

Lemma 4.1. *Let D be a finite direct sum of simple C*-algebras and let $\varphi, \psi : D \to A$ be $*$-homomorphisms. Suppose that $\mathcal{H} \subset D$ contains a nonzero element from each simple direct summand of D. If $\|\psi(d) - \varphi(d)\| \leq \|d\|/2$ for all $d \in \mathcal{H}$, then φ is injective if and only if ψ is injective.*

Proof. Let us note that φ is injective if and only if $\|\varphi(d)\| = \|d\|$ for all $d \in \mathcal{H}$. Therefore if φ is injective, then $\|\psi(d)\| \geq \|\varphi(d)\| - \|\psi(d) - \varphi(d)\| \geq \|d\|/2$ for all $d \in \mathcal{H}$ and hence ψ is injective. □

A sequence (A_n) of subalgebras of A is called *exhaustive* if for any finite subset $\mathcal{F}$ of A and any $\varepsilon > 0$ there is n such that $\mathcal{F} \subset_\varepsilon A_n$.

Lemma 4.2. *Let $\mathcal{C}$ be a class consisting of finite direct sums of separable simple weakly semiprojective C*-algebras. Let X be a compact metrizable space and let A be a $C(X)$-algebra. Let $\mathcal{F} \subset A$ be a finite set, let $\varepsilon > 0$ and suppose that $A(x)$ admits an exhaustive sequence of C*-algebras isomorphic to C*-algebras in $\mathcal{C}$ for some $x \in X$. Then there exist a compact neighborhood U of x and a $*$-homomorphism $\varphi : D \to A(U)$ for some $D \in \mathcal{C}$ such that $\pi_U(\mathcal{F}) \subset_\varepsilon \varphi(D)$. If A is a continuous $C(X)$-algebra, then we may arrange that φ_z is injective for all $z \in U$.*

Proof. Let $\mathcal{F} = \{a_1, \dots, a_r\}$ and ε be given. By hypothesis there exist $D \in \mathcal{C}$, $\{c_1, \dots, c_r\} \subset D$ and a $*$-monomorphism $\iota : D \to A(x)$ such that $\|\pi_x(a_i) - \iota(c_i)\| < \varepsilon/2$, for all $i = 1, ..., r$. Set $U_n = \{y \in X : d(x, y) \leq 1/n\}$. Choose a full element d_j in each direct summand of D. Since D is weakly semiprojective, there is a $*$-homomorphism $\varphi : D \to A(U_n)$ (for some n) such that $\|\pi_x\varphi(c_i) - \iota(c_i)\| < \varepsilon/2$ for all $i = 1, ..., r$, and $\|\pi_x\varphi(d_j) - \iota(d_j)\| \leq \|d_j\|/2$ for all d_j. Therefore

$$\|\pi_x\varphi(c_i) - \pi_x(a_i)\| \leq \|\pi_x\varphi(c_i) - \iota(c_i)\| + \|\pi_x(a_i) - \iota(c_i)\| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

and φ_x is injective by Lemma 4.1. By Lemma 2.1(i), after increasing n and setting $U = U_n$ and $\varphi = \pi_U\varphi$, we have

$$\|\varphi(c_i) - \pi_U(a_i)\| = \|\pi_U\big(\varphi(c_i) - a_i\big)\| < \varepsilon,$$

for all $i = 1, ..., r$. This shows that $\pi_U(\mathcal{F}) \subset_\varepsilon \varphi(D)$. If A is continuous, then after shrinking U we may arrange that $\|\varphi_z(d_j)\| \geq \|\varphi_x(d_j)\|/2 = \|d_j\|/2$ for all d_j and all $z \in U$. This implies that φ_z in injective for all $z \in U$. □

Lemma 4.3. *Let X be a compact metrizable space and let A be a separable continuous $C(X)$-algebra the fibers of which are stable Kirchberg algebras. Let $\mathcal{F} \subset A$ be a finite set and let $\varepsilon > 0$. Suppose that there exist a KK-semiprojective stable Kirchberg algebra D and $\sigma \in KK(D, A)$ such that $\sigma_x \in KK(D, A(x))^{-1}$ for some $x \in X$. Then there exist a closed neighborhood U of x and a full $*$-homomorphism $\psi : D \to A(U)$ such that $KK(\psi) = \sigma_U$ and $\pi_U(\mathcal{F}) \subset_\varepsilon \psi(D)$.*

Proof. By [29, Thm. 8.4.1] there is an isomorphism $\psi_0 : D \to A(x)$ such that $KK(\psi_0) = \sigma_x$. Let $\mathcal{H} \subset D$ be such that $\psi_0(\mathcal{H}) = \pi_x(\mathcal{F})$. Set $U_n = \{y \in X : d(x, y) \leq 1/n\}$. By Theorem 3.12 D is KK-stable and weakly semiprojective. By Proposition 3.7 there exists a $*$-homomorphism $\psi_n : D \to A(U_n)$ (for some n) such that $\|\pi_x\psi_n(d) - \psi_0(d)\| < \varepsilon$ for all $d \in \mathcal{H}$ and $KK(\pi_x\psi_n) =$

$KK(\psi_0)=\sigma_x$. Since $\varinjlim_m KK(D,A(U_m))=KK(D,A(x))$, we deduce that there is $m\geq n$ such that $KK(\pi_{U_m}\psi_n)=\sigma_{U_m}$. By increasing m we may arrange that $\pi_{U_m}(\mathcal{F})\subset_\varepsilon \pi_{U_m}\psi_n(D)$ since we have seen that $\pi_x(\mathcal{F})=\psi_0(\mathcal{H})\subset_\varepsilon \pi_x\psi_n(D)$. We can arrange that ψ_z is injective for all $z\in U$ by reasoning as in the proof of Lemma 4.2. We conclude by setting $U=U_m$ and $\psi=\pi_{U_m}\psi_n$. □

The following lemma is useful for constructing fibered morphisms.

Lemma 4.4. *Let $(D_j)_{j\in J}$ be a finite family consisting of finite direct sums of weakly semiprojective simple C*-algebras. Let $\varepsilon>0$ and for each $j\in J$ let $\mathcal{H}_j\subset D_j$ be a finite set such that for each direct summand of D_j there is an element of $\mathcal{H}_j$ of norm $\geq\varepsilon$ which is contained and is full in that summand. Let $\mathcal{G}_j\subset D_j$ and $\delta_j>0$ be given by Proposition 3.7 applied to D_j, $\mathcal{H}_j$ and $\varepsilon/2$. Let X be a compact metrizable space, let $(Z_j)_{j\in J}$ be disjoint nonempty closed subsets of X and let Y be a closed nonempty subset of X such that $X=Y\cup\big(\cup_j Z_j\big)$. Let A be a continuous $C(X)$-algebra and let $\mathcal{F}$ be a finite subset of A. Let $\eta: B(Y)\to A(Y)$ be a $*$-monomorphism of $C(Y)$-algebras and let $\varphi_j: D_j\to A(Z_j)$ be $*$-homomorphisms such that $(\varphi_j)_x$ is injective for all $x\in Z_j$ and $j\in J$, and which satisfy the following conditions:*

(i) $\pi_{Z_j}(\mathcal{F})\subset_{\varepsilon/2}\varphi_j(\mathcal{H}_j)$, *for all $j\in J$,*
(ii) $\pi_Y(\mathcal{F})\subset_\varepsilon \eta(B)$,
(iii) $\pi^{Z_j}_{Y\cap Z_j}\varphi_j(\mathcal{G}_j)\subset_{\delta_j}\pi^Y_{Y\cap Z_j}\eta(B)$, *for all $j\in J$.*

Then, there are $C(Z_j)$-linear $*$-monomorphisms $\psi_j: C(Z_j)\otimes D_j\to A(Z_j)$, satisfying

$$\|\varphi_j(c)-\psi_j(c)\|<\varepsilon/2,\ \text{for all } c\in\mathcal{H}_j,\ \text{and } j\in J, \tag{6}$$

and such that if we set $E=\bigoplus_j C(Z_j)\otimes D_j$, $Z=\cup_j Z_j$, and $\psi: E\to A(Z)=\bigoplus_j A(Z_j)$, $\psi=\oplus_j\psi_j$, then $\pi^Z_{Y\cap Z}(\psi(E))\subseteq\pi^Y_{Y\cap Z}(\eta(B))$, $\pi_Z(\mathcal{F})\subset_\varepsilon\psi(E)$ and hence

$$\mathcal{F}\subset_\varepsilon \eta(B)\oplus_{Y\cap Z}\psi(E)=\chi(B\oplus_{\pi\eta,\pi\psi}E),$$

where χ is the isomorphism induced by the pair (η,ψ). If we assume that each D_j is KK-stable, then we also have $KK(\varphi_j)=KK(\psi_j|_{D_j})$ for all $j\in J$.

Proof. Let $\mathcal{F}=\{a_1,\dots,a_r\}\subset A$ be as in the statement. By (i), for each $j\in J$ we find $\{c_1^{(j)},\dots,c_r^{(j)}\}\subseteq\mathcal{H}_j$ such that $\|\varphi_j(c_i^{(j)})-\pi_{Z_j}(a_i)\|<\varepsilon/2$ for all i. Consider the $C(X)$-algebra $A\oplus_Y\eta(B)\subset A$. From (iii), Lemma 2.1(iv) and Lemma 2.5 we obtain

$$\varphi_j(\mathcal{G}_j)\subset_{\delta_j}\pi_{Z_j}(A\oplus_Y\eta(B)).$$

Applying Proposition 3.7 we perturb φ_j to a $*$-homomorphism $\psi_j: D_j\to\pi_{Z_j}(A\oplus_Y\eta(B))$ satisfying (6), and hence such that $\|\varphi_j(c_i^{(j)})-\psi_j(c_i^{(j)})\|<\varepsilon/2$, for all i,j. Therefore

$$\|\psi_j(c_i^{(j)})-\pi_{Z_j}(a_i)\|\leq\|\psi_j(c_i^{(j)})-\varphi_j(c_i^{(j)})\|+\|\varphi_j(c_i^{(j)})-\pi_{Z_j}(a_i)\|<\varepsilon.$$

This shows that $\pi_{Z_j}(\mathcal{F})\subset_\varepsilon\psi_j(D_j)$. From (6) and Lemma 4.1 we obtain that each $(\psi_j)_x$ is injective. We extend ψ_j to a $C(Z_j)$-linear $*$-monomorphism $\psi_j: C(Z_j)\otimes D_j\to\pi_{Z_j}(A\oplus_Y\eta(B))$ and then we define E, ψ and Z as in the statement. In this way we obtain that $\psi: E\to(A\oplus_Y\eta(B))(Z)\subset A(Z)$ satisfies

$$\pi_Z(\mathcal{F})\subset_\varepsilon\psi(E). \tag{7}$$

The property $\psi(E)\subset(A\oplus_Y\eta(B))(Z)$ is equivalent to $\pi^Z_{Y\cap Z}(\psi(E))\subseteq\pi^Y_{Y\cap Z}(\eta(B))$ by Lemma 2.6(b). Finally, from (ii), (7) and Lemma 2.6 (c) we get $\mathcal{F}\subset_\varepsilon\eta(B)\oplus_{Y\cap Z}\psi(E)$. □

Let $\mathcal{C}$ be as in Lemma 4.2. Let A be a $C(X)$-algebra, let $\mathcal{F} \subset A$ be a finite set and let $\varepsilon > 0$. An $(\mathcal{F}, \varepsilon, \mathcal{C})$-*approximation* of A

$$\alpha = \{\mathcal{F}, \varepsilon, \{U_i, \varphi_i : D_i \to A(U_i), \mathcal{H}_i, \mathcal{G}_i, \delta_i\}_{i\in I}\}, \tag{8}$$

is a collection with the following properties: $(U_i)_{i\in I}$ is a finite family of closed subsets of X, whose interiors cover X and $(D_i)_{i\in I}$ are C*-algebras in $\mathcal{C}$; for each $i \in I$, $\varphi_i : D_i \to A(U_i)$ is a $*$-homomorphism such that $(\varphi_i)_x$ is injective for all $x \in U_i$; $\mathcal{H}_i \subset D_i$ is a finite set such that $\pi_{U_i}(\mathcal{F}) \subset_{\varepsilon/2} \varphi_i(\mathcal{H}_i)$ and such that for each direct summand of D_i there is an element of $\mathcal{H}_i$ of norm $\geq \varepsilon$ which is contained and is full in that summand; the finite set $\mathcal{G}_i \subset D_i$ and $\delta_i > 0$ are given by Proposition 3.7 applied to the weakly semiprojective C*-algebra D_i for the input data $\mathcal{H}_i$ and $\varepsilon/2$; if D_i is KK-stable, then $\mathcal{G}_i$ and δ_i are chosen such that the second part of Proposition 3.7 also applies.

Lemma 4.5. *Let A and $\mathcal{C}$ be as in Lemma 4.2. Suppose that each fiber of A admits an exhaustive sequence of C*-algebras isomorphic to C*-algebras in $\mathcal{C}$. Then for any finite subset $\mathcal{F}$ of A and any $\varepsilon > 0$ there is an $(\mathcal{F}, \varepsilon, \mathcal{C})$-approximation of A. Moreover, if A, D and σ are as in Lemma 4.3 and $\sigma_x \in KK(D, A(x))^{-1}$ for all $x \in X$, then there is an $(\mathcal{F}, \varepsilon, \mathcal{C})$-approximation of A such that $\mathcal{C} = \{D\}$ and $KK(\varphi_i) = \sigma_{U_i}$ for all $i \in I$.*

Proof. Since X is compact, this is an immediate consequence of Lemmas 4.2, 4.3 and Proposition 3.7. □

It is useful to consider the following operation of restriction. Suppose that Y is a closed subspace of X and let $(V_j)_{j\in J}$ be a finite family of closed subsets of Y which refines the family $(Y \cap U_i)_{i\in I}$ and such that the interiors of the V_j's form a cover of Y. Let $\iota : J \to I$ be a map such that $V_j \subseteq Y \cap U_{\iota(j)}$. Define

$$\iota^*(\alpha) = \{\pi_Y(\mathcal{F}), \varepsilon, \{V_j, \pi_{V_j}\varphi_{\iota(j)} : D_{\iota(j)} \to A(V_j), \mathcal{H}_{\iota(j)}, \mathcal{G}_{\iota(j)}, \delta_{\iota(j)}\}_{j\in J}\}.$$

It is obvious that $\iota^*(\alpha)$ is a $(\pi_Y(\mathcal{F}), \varepsilon, \mathcal{C})$-approximation of $A(Y)$. The operation $\alpha \mapsto \iota^*(\alpha)$ is useful even in the case $X = Y$. Indeed, by applying this procedure we can refine the cover of X that appears in a given $(\mathcal{F}, \varepsilon, \mathcal{C})$-approximation of A.

An $(\mathcal{F}, \varepsilon, \mathcal{C})$-approximation α (as in (8)) is subordinated to an $(\mathcal{F}', \varepsilon', \mathcal{C})$-approximation, $\alpha' = \{\mathcal{F}', \varepsilon', \{U_{i'}, \varphi_{i'} : D_{i'} \to A(U_{i'}), \mathcal{H}_{i'}, \mathcal{G}_{i'}, \delta_{i'}\}_{i'\in I'}\}$, written $\alpha \prec \alpha'$, if

(i) $\mathcal{F} \subseteq \mathcal{F}'$,
(ii) $\varphi_i(\mathcal{G}_i) \subseteq \pi_{U_i}(\mathcal{F}')$ for all $i \in I$, and
(iii) $\varepsilon' < \min\big(\{\varepsilon\} \cup \{\delta_i,\, i \in I\}\big)$.

Let us note that, with notation as above, we have $\iota^*(\alpha) \prec \iota^*(\alpha')$ whenever $\alpha \prec \alpha'$.

The following theorem is the crucial technical result of our paper. It provides an approximation of continuous $C(X)$-algebras by subalgebras of category $\leq \dim(X)$.

Theorem 4.6. *Let $\mathcal{C}$ be a class consisting of finite direct sums of weakly semiprojective simple C*-algebras. Let X be a finite dimensional compact metrizable space and let A be a separable continuous $C(X)$-algebra the fibers of which admit exhaustive sequences of C*-algebras isomorphic to C*-algebras in $\mathcal{C}$. For any finite set $\mathcal{F} \subset A$ and any $\varepsilon > 0$ there exist $n \leq \dim(X)$ and an n-fibered $\mathcal{C}$-monomorphism $(\psi_0, \ldots, \psi_n)$ into A which induces a $*$-monomorphism $\eta : A(\psi_0, ..., \psi_n) \to A$ such that $\mathcal{F} \subset_\varepsilon \eta(A(\psi_0, ..., \psi_n))$.*

Proof. By Lemma 4.5, for any finite set $\mathcal{F} \subset A$ and any $\varepsilon > 0$ there is an $(\mathcal{F}, \varepsilon, \mathcal{C})$-approximation of A. Moreover, for any finite set $\mathcal{F} \subset A$, any $\varepsilon > 0$ and any n, there is a sequence $\{\alpha_k : 0 \leq k \leq n\}$ of $(\mathcal{F}_k, \varepsilon_k, \mathcal{C})$-approximations of A such that $(\mathcal{F}_0, \varepsilon_0) = (\mathcal{F}, \varepsilon)$ and α_k is subordinated to α_{k+1}:

$$\alpha_0 \prec \alpha_1 \prec \cdots \prec \alpha_n.$$

Indeed, assume that α_k was constructed. Let us choose a finite set $\mathcal{F}_{k+1}$ which contains $\mathcal{F}_k$ and liftings to A of all the elements in $\bigcup_{i_k \in I_k} \varphi_{i_k}(\mathcal{G}_{i_k})$. This choice takes care of the above conditions (i) and (ii). Next we choose ε_{k+1} sufficiently small such that (iii) is satisfied. Let α_{k+1} be an $(\mathcal{F}_{k+1}, \varepsilon_{k+1}, \mathcal{C})$-approximation of A given by Lemma 4.5. Then obviously $\alpha_k \prec \alpha_{k+1}$. Fix a tower of approximations of A as above where $n = \dim(X)$.

By [4, Lemma 3.2], for every open cover $\mathcal{V}$ of X there is a finite open cover $\mathcal{U}$ which refines $\mathcal{V}$ and such that the set $\mathcal{U}$ can be partitioned into $n+1$ nonempty subsets consisting of elements with pairwise disjoint closures. Since we can refine simultaneously the covers that appear in a finite family $\{\alpha_k : 0 \leq k \leq n\}$ of approximations while preserving subordination, we may arrange not only that all α_k share the same cover $(U_i)_{\in I}$, but moreover, that the cover $(U_i)_{i \in I}$ can be partitioned into $n+1$ subsets $\mathcal{U}_0, \ldots, \mathcal{U}_n$ consisting of mutually disjoint elements. For definiteness, let us write $\mathcal{U}_k = \{U_{i_k} : i_k \in I_k\}$. Now for each k we consider the closed subset of X

$$Y_k = \bigcup_{i_k \in I_k} U_{i_k},$$

the map $\iota_k : I_k \to I$ and the $(\pi_{Y_k}(\mathcal{F}_k), \varepsilon_k, \mathcal{C})$-approximation of $A(Y_k)$, induced by α_k, which is of the form

$$\iota_k^*(\alpha_k) = \{\pi_{Y_k}(\mathcal{F}_k), \varepsilon, \{U_{i_k}, \varphi_{i_k} : D_{i_k} \to A(U_{i_k}), \mathcal{H}_{i_k}, \mathcal{G}_{i_k}, \delta_{i_k}\}_{i_k \in I_k}\},$$

where each U_{i_k} is nonempty. We have

$$\pi_{U_{i_k}}(\mathcal{F}_k) \subset_{\varepsilon_k/2} \varphi_{i_k}(\mathcal{H}_{i_k}), \tag{9}$$

by construction. Since $\alpha_k \prec \alpha_{k+1}$ we obtain

$$\mathcal{F}_k \subseteq \mathcal{F}_{k+1}, \tag{10}$$

$$\varphi_{i_k}(\mathcal{G}_{i_k}) \subseteq \pi_{U_{i_k}}(\mathcal{F}_{k+1}), \text{ for all } i_k \in I_k, \tag{11}$$

$$\varepsilon_{k+1} < \min\left(\{\varepsilon_k\} \cup \{\delta_{i_k},\, i_k \in I_k\}.\right) \tag{12}$$

Set $X_k = Y_k \cup \cdots \cup Y_n$ and $E_k = \oplus_{i_k} C(U_{i_k}) \otimes D_{i_k}$ for $0 \leq k \leq n$. We shall construct a sequence of $C(Y_k)$-linear $*$-monomorphisms, $\psi_k : E_k \to A(Y_k)$, $k = n, ..., 0$, such that $(\psi_k, \ldots, \psi_n)$ is an $(n-k)$-fibered monomorphism into $A(X_k)$. Each map

$$\psi_k = \oplus_{i_k} \psi_{i_k} : E_k \to A(Y_k) = \oplus_{i_k} A(U_{i_k})$$

will have components $\psi_{i_k} : C(U_{i_k}) \otimes D_{i_k} \to A(U_{i_k})$ whose restrictions to D_{i_k} will be perturbations of $\varphi_{i_k} : D_{i_k} \to A(U_{i_k})$, $i_k \in I_k$. We shall construct the maps ψ_k by induction on decreasing k such that if $B_k = A(X_k)(\psi_k, \ldots, \psi_n)$ and $\eta_k : B_k \to A(X_k)$ is the map induced by the $(n-k)$-fibered monomorphism $(\psi_k, \ldots, \psi_n)$, then

$$\pi_{X_{k+1} \cap U_{i_k}}\big(\psi_{i_k}(D_{i_k})\big) \subset \pi_{X_{k+1} \cap U_{i_k}}\big(\eta_{k+1}(B_{k+1})\big), \, \forall\, i_k \in I_k, \tag{13}$$

and

$$\pi_{X_k}(\mathcal{F}_k) \subset_{\varepsilon_k} \eta_k(B_k). \tag{14}$$

Note that (13) is equivalent to

$$\pi_{X_{k+1}\cap Y_k}\big(\psi_k(E_k)\big) \subset \pi_{X_{k+1}\cap Y_k}\big(\eta_{k+1}(B_{k+1})\big). \tag{15}$$

For the first step of induction, $k = n$, we choose $\psi_n = \oplus_{i_n}\widetilde{\varphi}_{i_n}$ where $\widetilde{\varphi}_{i_n} : C(U_{i_n}) \otimes D_{i_n} \to A(U_{i_n})$ are $C(U_{i_n})$-linear extensions of the original φ_{i_n}. Then $B_n = E_n$ and $\eta_n = \psi_n$. Assume that $\psi_n, \dots, \psi_{k+1}$ were constructed and that they have the desired properties. We shall construct now ψ_k. Condition (14) formulated for $k + 1$ becomes

$$\pi_{X_{k+1}}(\mathcal{F}_{k+1}) \subset_{\varepsilon_{k+1}} \eta_{k+1}(B_{k+1}). \tag{16}$$

Since $\varepsilon_{k+1} < \delta_{i_k}$, by using (11) and (16) we obtain

$$\pi_{X_{k+1}\cap U_{i_k}}\big(\varphi_{i_k}(\mathcal{G}_{i_k})\big) \subset_{\delta_{i_k}} \pi_{X_{k+1}\cap U_{i_k}}\big(\eta_{k+1}(B_{k+1})\big), \text{ for all } i_k \in I_k. \tag{17}$$

Since $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$ and $\varepsilon_{k+1} < \varepsilon_k$, condition (16) gives

$$\pi_{X_{k+1}}(\mathcal{F}_k) \subset_{\varepsilon_k} \eta_{k+1}(B_{k+1}). \tag{18}$$

Conditions (9), (17) and (18) enable us to apply Lemma 4.4 and perturb $\widetilde{\varphi}_{i_k}$ to a $*$-monomorphism $\psi_{i_k} : C(U_{i_k}) \otimes D_{i_k} \to A(U_{i_k})$ satisfying (13) and (14) and such that

$$KK(\psi_{i_k}|_{D_{i_k}}) = KK(\varphi_{i_k}) \tag{19}$$

if the algebras in $\mathcal{C}$ are assumed to be KK-stable. We set $\psi_k = \oplus_{i_k} \psi_{i_k}$ and this completes the construction of $(\psi_0, \dots, \psi_n)$. Condition (14) for $k = 0$ gives $\mathcal{F} \subset_\varepsilon \eta_0(B_0) = \eta(A(\psi_0, \dots, \psi_n))$. Thus $(\psi_0, \dots, \psi_n)$ satisfies the conclusion of the theorem. Finally let us note that it can happen that $X_k = X$ for some $k > 0$. In this case $\mathcal{F} \subset_\varepsilon A(\psi_k, ..., \psi_n)$ and for this reason we write $n \leq \dim(X)$ in the statement of the theorem. □

Proposition 4.7. *Let X be a finite dimensional compact metrizable space and let A be a separable continuous $C(X)$-algebra the fibers of which are stable Kirchberg algebras. Let D be a KK-semiprojective stable Kirchberg algebra and suppose that there exists $\sigma \in KK(D, A)$ such that $\sigma_x \in KK(D, A(x))^{-1}$ for all $x \in X$. For any finite subset $\mathcal{F}$ of A and any $\varepsilon > 0$ there is an n-fibered $\mathcal{C}$-monomorphism $(\psi_0, \dots, \psi_n)$ into A such that $n \leq \dim(X)$, $\mathcal{C} = \{D\}$, and each component $\psi_i : C(Y_i) \otimes D \to A(Y_i)$ satisfies $KK(\psi_i) = \sigma_{Y_i}$, $i = 0, \dots, n$. Moreover, if $\eta : A(\psi_0, \dots, \psi_n) \to A$ is the induced $*$-monomorphism, then $\mathcal{F} \subset_\varepsilon \eta(A(\psi_0, \dots, \psi_n))$ and $KK(\eta_x)$ is a KK-equivalence for each $x \in X$.*

Proof. We repeat the proof of Theorem 4.6 while using only $(\mathcal{F}_i, \varepsilon_i, \{D\})$-approximations of A provided by the second part of Lemma 4.5. The outcome will be an n-fibered $\{D\}$-monomorphism $(\psi_0, \dots, \psi_n)$ into A such that $\mathcal{F} \subset_\varepsilon \eta(A(\psi_0, \dots, \psi_n))$. Moreover we can arrange that $KK(\psi_i) = \sigma_{Y_i}$ for all $i = 0, \dots, n$, by (19), since $KK(\varphi_{i_k}) = \sigma_{U_{i_k}}$ by Lemma 4.5. If $x \in X$, and $i = \min\{k : x \in Y_k\}$, then $\eta_x \equiv (\psi_i)_x$, and hence $KK(\eta_x)$ is a KK-equivalence. □

Remark 4.8. Let us point out that we can strengthen the conclusion of Theorem 4.6 and Proposition 4.7 as follows. Fix a metric d for the topology of X. Then we may arrange that there is a closed cover $\{Y_0', ..., Y_n'\}$ of X and a number $\ell > 0$ such that $\{x : d(x, Y_i') \leq \ell\} \subset Y_i$ for $i = 0, ..., n$.

Indeed, when we choose the finite closed cover $\mathcal{U} = (U_i)_{i\in I}$ of X in the proof of Theorem 4.6 which can be partitioned into $n+1$ subsets $\mathcal{U}_0, \dots, \mathcal{U}_n$ consisting of mutually disjoints elements, as given by [4, Lemma 3.2], and which refines all the covers $\mathcal{U}(\alpha_0), ..., \mathcal{U}(\alpha_n)$ corresponding to $\alpha_0, ..., \alpha_n$, we may assume that $\mathcal{U}$ also refines the covers given by the interiors of the elements of $\mathcal{U}(\alpha_0), ..., \mathcal{U}(\alpha_n)$. Since each U_i is compact and I is finite, there is $\ell > 0$ such that if $V_i = \{x : d(x, U_i) \leq \ell\}$, then the cover $\mathcal{V} = (V_i)_{i\in I}$ still refines all of $\mathcal{U}(\alpha_0), ..., \mathcal{U}(\alpha_n)$ and for each $k = 0, ..., n$, the elements of $\mathcal{V}_k = \{V_i : U_i \in \mathcal{U}_k\}$, are still mutually disjoint. We shall use the cover $\mathcal{V}$ rather than $\mathcal{U}$ in the proof of the two theorems and observe that $Y_k' \overset{def}{=} \bigcup_{i_k\in I_k} U_{i_k} \subset \bigcup_{i_k\in I_k} V_{i_k} = Y_k$ has the desired property. Finally let us note that if we define $\psi_i' : E(Y_i') \to A(Y_i')$ by $\psi_i' = \pi_{Y_i'}\psi_i$, then $(\psi_0', \dots, \psi_n')$ is an n-fibered $\mathcal{C}$-monomorphism into A which satisfies the conclusion of Theorem 4.6 and Proposition 4.7 since $\pi_{Y_i'}(\mathcal{F}) \subset_\varepsilon \psi_i'(E_i)$ for all $i = 0, \dots, n$ and $X = \bigcup_{i=1}^n Y_i'$.

5. Representing $C(X)$-algebras as inductive limits

We have seen that Theorem 4.6 yields exhaustive sequences for certain $C(X)$-algebras. In this section we show how to pass from an exhaustive sequence to a nested exhaustive sequence using semiprojectivity. The remainder of the paper does not depend on this section.

Proposition 5.1. *Let X, A and $\mathcal{C}$ be as in Theorem 4.6. Let $(\psi_0, \dots, \psi_n)$ be an n-fibered $\mathcal{C}$-monomorphism into A with components $\psi_i : E_i \to A(Y_i)$. Let $\mathcal{F}_i \subset E_i$, $\mathcal{F} \subset A(\psi_0, ..., \psi_n)$ be finite sets and let $\varepsilon > 0$. Then there are finite sets $\mathcal{G}_i \subset E_i$ and $\delta_i > 0$, $i = 0, ..., n$, such that for any $C(X)$-subalgebra $A' \subset A$ which satisfies $\psi_i(\mathcal{G}_i) \subset_{\delta_i} A'(Y_i)$, $i = 0, ..., n$, there is an n-fibered $\mathcal{C}$-monomorphism $(\psi_0', \dots, \psi_n')$ into A', with $\psi_i' : E_i \to A'(Y_i)$ and such that (i) $\|\psi_i(a) - \psi_i'(a)\| < \varepsilon$ for all $a \in \mathcal{F}_i$ and all $i \in \{0, ..., n\}$, (ii) $(\psi_j)_x^{-1}(\psi_i)_x = (\psi_j')_x^{-1}(\psi_i')_x$ for all $x \in Y_i \cap Y_j$ and $0 \leq i \leq j \leq n$. Moreover $A(\psi_0, \dots, \psi_n) = A'(\psi_0', \dots, \psi_n')$ and the maps $\eta : A(\psi_0, ..., \psi_n) \to A$ and $\eta' : A'(\psi_0', \dots, \psi_n') \to A'$ induced by $(\psi_0, \dots, \psi_n)$ and $(\psi_0', \dots, \psi_n')$ satisfy (iii) $\|\eta(a) - \eta'(a)\| < \varepsilon$ for all $a \in \mathcal{F}$.*

Proof. Let us observe that if we prove (i) and (ii) then (iii) will follow by enlarging the sets $\mathcal{F}_i$ so that $p_i(\mathcal{F}) \subset \mathcal{F}_i$, where $p_i : A(\psi_0, ..., \psi_n) \to E_i$ are the coordinate maps. We proceed now with the proof of (i) and (ii) by making some simplifications. We may assume that $E_0 = C(Y_0) \otimes D_0$ with $D_0 \in \mathcal{C}$ since the perturbations corresponding to disjoint closed sets can be done independently of each other. Without any loss of generality, we may assume that $\mathcal{F}_0 \subset D_0$ since we are working with morphisms on E_0 which are $C(Y_0)$-linear. We also enlarge $\mathcal{F}_0$ so that for each direct summand C of D_0, $\mathcal{F}_0$ contains an element c which is full in C and such that $\|c\| \geq 2\varepsilon$.

The proof is by induction on n. If $n = 0$ the statement follows from Proposition 3.7 and Lemma 4.1. Assume now that the statement is true for $n-1$. Let E_i, ψ_i, A, A', $\mathcal{F}_i$, $1 \leq i \leq n$ and ε be as in the statement. For $0 \leq i < j \leq n$ let $\eta_{j,i} : E_i(Y_i \cap Y_j) \to E_j(Y_i \cap Y_j)$ be the $*$-homomorphism of $C(Y_i \cap Y_j)$-algebras defined fiberwise by $(\eta_{j,i})_x = (\psi_j^{-1})_x(\psi_i)_x$

Let $\mathcal{G}_0$ and δ_0 be given by Proposition 3.8 applied to the C*-algebra D_0 for the input data $\mathcal{F}_0$ and ε. For each $1 \leq j \leq n$ choose a finite subset $\mathcal{H}_j$ of E_j whose restriction to $Y_j \cap Y_0$ contains $\eta_{j,0}(\mathcal{G}_0)$. Consider the sets $\mathcal{F}_j' := \mathcal{F}_j \cup \mathcal{H}_j$, $1 \leq j \leq n$ and the number $\varepsilon' = \min\{\delta_0, \varepsilon\}$. Let $\mathcal{G}_1, ...\mathcal{G}_n$ and $\delta_1, ..., \delta_n$ be given by the inductive assumption for $n-1$ applied to $A(X_1)$, $A'(X_1)$, ψ_j, $\mathcal{F}_j'$, $1 \leq j \leq n$ and ε', where $X_1 = Y_1 \cup \dots \cup Y_n$.

We need to show that $\mathcal{G}_0, \mathcal{G}_1, ...\mathcal{G}_n$ and $\delta_0, \delta_1, ..., \delta_n$ satisfy the statement. By the inductive step there exists an $(n-1)$-fibered $\mathcal{C}$-monomorphism $(\psi'_1, \dots, \psi'_n)$ into $A'(X_1)$ with components $\psi'_j : E_j \to A'(Y_j)$ such that

(a) $\|\psi_j(a) - \psi'_j(a)\| < \varepsilon' = \min\{\delta_0, \varepsilon\}$ for all $a \in \mathcal{F}_j \cup \mathcal{H}_j$ and all $1 \leq j \leq n$,

(b) $(\psi_j)_x^{-1}(\psi_i)_x = (\psi'_j)_x^{-1}(\psi'_i)_x$ for all $x \in Y_i \cap Y_j$ and $1 \leq i \leq j \leq n$,

The condition (b) enables to define a $*$-homomorphism $\varphi : E_0 \to A'(Y_0 \cap X_1)$ with fiber maps $\varphi_x = (\psi'_j)_x(\psi_j^{-1})_x(\psi_0)_x$ for $x \in Y_0 \cap Y_j$ and $1 \leq j \leq n$.

Let us observe that $\psi_0 : E_0 \to A(Y_0)$ is an approximate lifting of φ. More precisely we have $\|\pi^{Y_0}_{X_1 \cap Y_0}\psi_0(a) - \varphi(a)\| < \delta_0$ for all $a \in \mathcal{G}_0$. Indeed, for $x \in Y_0 \cap Y_j$, $1 \leq j \leq n$ and $a \in \mathcal{G}_0$ we have

$$\begin{aligned}\|(\psi_0)_x(a(x)) - (\psi'_j)_x(\psi_j^{-1})_x(\psi_0)_x(a(x))\| &= \|(\psi_j)_x(\eta_{j,0})_x(a(x)) - (\psi'_j)_x(\eta_{j,0})_x(a(x))\| \\ &\leq \sup_{h \in \mathcal{H}_j} \|\psi_j(h) - \psi'_j(h)\| < \varepsilon' \leq \delta_0.\end{aligned}$$

Since we also have $\psi_0(\mathcal{G}_0) \subset_{\delta_0} A'(Y_0)$ by hypothesis, it follows from Proposition 3.8 that there exists $\psi'_0 : D_0 \to A(Y_0)$ such that $\|\psi'_0(a) - \psi_0(a)\| < \varepsilon$ for all $a \in \mathcal{F}_0$ and $\pi^{Y_0}_{Y_0 \cap X_1}\psi'_0 = \varphi$. By Lemma 4.1 each $(\psi'_0)_x$ is injective since each $(\psi_0)_x$ is injective. The $C(Y_0)$-linear extension of ψ'_0 to E_0 satisfies $(\psi_j)_x^{-1}(\psi_0)_x = (\psi'_j)_x^{-1}(\psi'_0)_x$ for all $x \in Y_0 \cap Y_j$ and $1 \leq j \leq n$ and this completes the proof of (ii). Condition (i) follows from (b). □

The following result gives an inductive limit representation for continuous $C(X)$-algebras whose fibers are inductive limits of finite direct sums of simple semiprojective C*-algebras. For example the fibers can be arbitrary AF-algebras or Kirchberg algebras which satisfy the UCT and whose K_1-groups are torsion free. Indeed, by [29, Prop. 8.4.13], these algebras are isomorphic to inductive limits of sequences of Kirchberg algebras (D_n) with finitely generated K-theory groups and torsion free K_1-groups. The algebras D_n are semiprojective by [33].

Theorem 5.2. *Let $\mathcal{C}$ be a class consisting of finite direct sums of semiprojective simple C*-algebras. Let X be a finite dimensional compact metrizable space and let A be a separable continuous $C(X)$-algebra such that all its fibers admit exhaustive sequences consisting of C*-algebras isomorphic to C*-algebras in $\mathcal{C}$. Then A is isomorphic to the inductive limit of a sequence of continuous $C(X)$-algebras A_k such that $\mathrm{cat}_{\mathcal{C}}(A_k) \leq \dim(X)$.*

Proof. By Theorem 4.6 and Proposition 5.1 we find a sequence $(\psi_0^{(k)}, ..., \psi_n^{(k)})$ of n-fibered $\mathcal{C}$-monomorphisms into A which induces $*$-monomorphisms $\eta^{(k)} : A_k = A(\psi_0^{(k)}, ..., \psi_n^{(k)}) \to A$ with the following properties. There is a sequence of finite sets $\mathcal{F}_k \subset A_k$ and a sequence of $C(X)$-linear $*$-monomorphisms $\mu_k : A_k \to A_{k+1}$ such that

(i) $\|\eta^{(k+1)}\mu_k(a) - \eta^{(k)}(a)\| < 2^{-k}$ for all $a \in \mathcal{F}_k$ and all $k \geq 1$,

(ii) $\mu_k(\mathcal{F}_k) \subset \mathcal{F}_{k+1}$ for all $k \geq 1$,

(iii) $\bigcup_{j=k+1}^{\infty} \left(\mu_{j-1} \circ \cdots \circ \mu_k\right)^{-1}(\mathcal{F}_j)$ is dense in A_k and $\bigcup_{j=k}^{\infty} \eta^{(j)}(\mathcal{F}_j)$ is dense in A for all $k \geq 1$.

Arguing as in the proof of [29, Prop. 2.3.2], one verifies that

$$\varphi_k(a) = \lim_{j \to \infty} \eta^{(j)} \circ \left(\mu_{j-1} \circ \cdots \circ \mu_k\right)(a)$$

defines a sequence of $*$-monomorphisms $\varphi_k : A_k \to A$ such that $\varphi_{k+1}\mu_k = \varphi_k$ and the induced map $\varphi : \varinjlim_k (A_k, \mu_k) \to A$ is an isomorphism of $C(X)$-algebras. □

Remark 5.3. By similar arguments one proves a unital version of Theorem 5.2.

6. When is a fibered product locally trivial

For C*-algebras A, B we endow the space $\mathrm{Hom}(A,B)$ of $*$-homomorphisms with the point-norm topology. If X is a compact Hausdorff space, then $\mathrm{Hom}(A, C(X)\otimes B)$ is homeomorphic to the space of continuous maps from X to $\mathrm{Hom}(A,B)$ endowed with the compact-open topology. We shall identify a $*$-homomorphism $\varphi \in \mathrm{Hom}(A, C(X)\otimes B)$ with the corresponding continuous map $X \to \mathrm{Hom}(A,B)$, $x \mapsto \varphi_x$, $\varphi_x(a) = \varphi(a)(x)$ for all $x \in X$ and $a \in A$. Let D be a C*-algebra and let A be a $C(X)$-algebra. If $\alpha : D \to A$ is a $*$-homomorphism, let us denote by $\widetilde{\alpha} : C(X)\otimes D \to A$ its (unique) $C(X)$-linear extension and write $\widetilde{\alpha} \in \mathrm{Hom}_{C(X)}(C(X)\otimes D, A)$. For C*-algebras D, B we shall make without further comment the following identifications

$$\mathrm{Hom}_{C(X)}(C(X)\otimes D, C(X)\otimes B) \equiv \mathrm{Hom}(D, C(X)\otimes B) \equiv C(X, \mathrm{Hom}(D,B)).$$

For a C*-algebra D we denote by $\mathrm{End}(D)$ the set of full (*and unital if D is unital*) $*$-endomorphisms of D and by $\mathrm{End}(D)^0$ the path component of id_D in $\mathrm{End}(D)$. Let us consider

$$\mathrm{End}(D)^* = \{\gamma \in \mathrm{End}(D) : KK(\gamma) \in KK(D,D)^{-1}\}.$$

Proposition 6.1. *Let X be a compact metrizable space and let D be a KK-semiprojective Kirchberg algebra. Let $\alpha : D \to C(X)\otimes D$ be a full (and unital, if D is unital) $*$-homomorphism such that $KK(\alpha_x) \in KK(D,D)^{-1}$ for all $x \in X$. Then there is a full $*$-homomorphism $\Phi : D \to C(X\times[0,1])\otimes D$ such that $\Phi_{(x,0)} = \alpha_x$ and $\Phi_{(x,t)} \in \mathrm{Aut}(D)$ for all $x \in X$ and $t \in (0,1]$. Moreover, if $\Phi_1 : D \to C(X)\otimes D$ is defined by $\Phi_1(d)(x) = \Phi_{(x,1)}(d)$, for all $d \in D$ and $x \in X$, then $\alpha \approx_{uh} \Phi_1$.*

Proof. Since X is a metrizable compact space, X is homeomorphic to the projective limit of a sequence of finite simplicial complexes (X_i) by [13, Thm. 10.1, p.284]. Since D is KK-semiprojective, $KK(D, \varinjlim C(X_i)\otimes D) = KK(D, C(X)\otimes D)$ by Theorem 3.12. By Theorem 3.1, there is i and a full (and unital if D is unital) $*$-homomorphism $\varphi : D \to C(X_i)\otimes D$ whose KK-class maps to $KK(\alpha) \in KK(D, C(X)\otimes D)$. To summarize, we have found a finite simplicial complex Y, a continuous map $h : X \to Y$ and a continuous map $y \mapsto \varphi_y \in \mathrm{End}(D)$, defined on Y, such that the full (and unital if D is unital) $*$-homomorphism $h^*\varphi : D \to C(X)\otimes D$ corresponding to the continuous map $x \mapsto \varphi_{h(x)}$ satisfies $KK(h^*\varphi) = KK(\alpha)$. We may arrange that $h(X)$ intersects all the path components of Y by dropping the path components which are not intersected. Since $\alpha_x \in \mathrm{End}(D)^*$ by hypothesis, and since $KK(\alpha_x) = KK(\varphi_{h(x)})$, we infer that $\varphi_y \in \mathrm{End}(D)^*$ for all $y \in Y$. We shall find a continuous map $y \mapsto \psi_y \in \mathrm{End}(D)^*$ defined on Y, such that the maps $y \mapsto \psi_y\varphi_y$ and $y \mapsto \varphi_y\psi_y$ are homotopic to the constant map ι that takes Y to id_D. It is clear that it suffices to deal separately with each path component of Y, so that for this part of the proof we may assume that Y is connected. Fix a point $z \in Y$. By [29, Thm. 8.4.1] there is $\nu \in \mathrm{Aut}(D)$ such that $KK(\nu^{-1}) = KK(\varphi_z)$ and hence $KK(\nu\varphi_z) = KK(\mathrm{id}_D)$. By Theorem 3.1, there is a unitary $u \in M(D)$ such that $u\nu\varphi_z(-)u^*$ is homotopic to id_D. Let us set $\theta = u\nu(-)u^* \in \mathrm{Aut}(D)$ and observe that $\theta\varphi_z \in \mathrm{End}(D)^0$. Since Y is path connected, it follows that the entire image of the map $y \mapsto \theta\varphi_y$ is contained in $\mathrm{End}(D)^0$. Since $\mathrm{End}(D)^0$ is a path connected H-space with unit element, it follows by [38, Thm. 2.4, p462] that the homotopy classes $[Y, \mathrm{End}(D)^0]$ (with no condition on basepoints, since the action of the fundamental group $\pi_1(\mathrm{End}(D)^0, \mathrm{id}_D)$ is trivial by [38, 3.6, p166]) form a group under the natural multiplication. Therefore we find $y \mapsto \psi'_y \in \mathrm{End}(D)^0$ such that $y \mapsto \psi'_y \theta \varphi_y$ and $y \mapsto \theta \varphi_y \psi'_y$ are homotopic to ι. It follows that $y \mapsto \psi_y \overset{def}{=} \psi'_y \theta$ is the homotopic inverse of $y \mapsto \varphi_y$ in $[Y, \mathrm{End}(D)^*]$. Composing with h we obtain that the maps $x \mapsto \varphi_{h(x)} \psi_{h(x)}$ and $x \mapsto \psi_{h(x)} \varphi_{h(x)}$ are homotopic to the constant map that takes X to id_D. By the homotopy invariance of KK-theory we obtain that

$$KK(\widetilde{h^*\varphi}\, h^*\psi) = KK(\widetilde{h^*\psi}\, h^*\varphi) = KK(\iota_D),$$

where $\widetilde{h^*\varphi}$ and $\widetilde{h^*\psi}$ denote the $C(X)$-linear extensions of the corresponding maps and $\iota_D : D \to C(X) \otimes D$ is defined by $\iota_D(d) = 1_{C(X)} \otimes d$ for all $d \in D$. Let us recall that $KK(h^*\varphi) = KK(\alpha)$ and hence $KK(\widetilde{h^*\varphi}) = KK(\widetilde{\alpha})$. If we set $\Psi = h^*\psi$, then

$$KK(\widetilde{\alpha}\, \Psi) = KK(\widetilde{\Psi}\, \alpha) = KK(\iota_D).$$

By Theorem 3.1 $\widetilde{\alpha}\, \Psi \approx_u \iota_D$ and $\widetilde{\Psi}\, \alpha \approx_u \iota_D$, and hence $\widetilde{\alpha}\, \widetilde{\Psi} \approx_u \mathrm{id}_{C(X)\otimes D}$ and $\widetilde{\Psi}\, \widetilde{\alpha} \approx_u \mathrm{id}_{C(X)\otimes D}$. By [29, Cor. 2.3.4], there is an isomorphism $\Gamma : C(X)\otimes D \to C(X)\otimes D$ such that $\Gamma \approx_u \widetilde{\alpha}$. In particular Γ is $C(X)$-linear and $\Gamma_x \in \mathrm{Aut}(D)$ for all $x \in X$. Replacing Γ by $u\Gamma(\cdot)u^*$ for some unitary $u \in M(C(X)\otimes D)$ we can arrange that $\Gamma|_D$ is arbitrarily close to α. Therefore $KK(\Gamma|_D) = KK(\alpha)$ since D is KK-stable. By Theorem 3.1 there is a continuous map $(0,1] \to U(M(C(X)\otimes D))$, $t \mapsto u_t$, with the property that

$$\lim_{t\to 0} \|u_t \Gamma(a) u_t^* - \alpha(a)\| = 0, \text{ for all } \quad a \in D.$$

Therefore the equation

$$\Phi_{(x,t)} = \begin{cases} \alpha_x, & \text{if } t = 0, \\ u_t(x)\Gamma_x u_t(x)^*, & \text{if } t \in (0,1], \end{cases}$$

defines a continuous map $\Phi : X \times [0,1] \to \mathrm{End}(D)^*$ which extends α and such that $\Phi(X \times (0,1]) \subset \mathrm{Aut}(D)$. Since α is homotopic to Φ_1, we have that $\alpha \approx_{uh} \Phi_1$ by Theorem 3.1. □

Proposition 6.2. *Let X be a compact metrizable space and let D be a KK-semiprojective Kirchberg algebra. Let Y be a closed subset of X. Assume that a map $\gamma : Y \to \mathrm{End}(D)^*$ extends to a continuous map $\alpha : X \to \mathrm{End}(D)^*$. Then there is a continuous extension $\eta : X \to \mathrm{End}(D)^*$ of γ, such that $\eta(X \setminus Y) \subset \mathrm{Aut}(D)$.*

Proof. Since the map $x \to \alpha_x$ takes values in $\mathrm{End}(D)^*$, by Proposition 6.1 there exists a continuous map $\Phi : X \times [0,1] \to \mathrm{End}(D)^*$ which extends α and such that $\Phi(X \times (0,1]) \subset \mathrm{Aut}(D)$. Let d be a metric for the topology of X such that $\mathrm{diam}(X) \le 1$. The equation $\eta(x) = \Phi(x, d(x,Y))$ defines a map on X that satisfies the conclusion of the proposition. □

Lemma 6.3. *Let X be a compact metrizable space and let D be a KK-semiprojective Kirchberg algebra. Let Y be a closed subset of X. Let $\alpha : Y \times [0,1] \cup X \times \{0\} \to \mathrm{End}(D)$ be a continuous map such that $\alpha_{(x,0)} \in \mathrm{End}(D)^*$ for all $x \in X$. Suppose that there is an open set V in X which contains Y and such that α extends to a continuous map $\alpha_V : V \times [0,1] \cup X \times \{0\} \to \mathrm{End}(D)$. Then there is $\eta : X \times [0,1] \to \mathrm{End}(D)^*$ such that η extends α and $\eta_{(x,t)} \in \mathrm{Aut}(D)$ for all $x \in X \setminus Y$ and $t \in (0,1]$.*

Proof. By Proposition 6.2 it suffices to find a continuous map $\widehat{\alpha} : X \times [0,1] \to \mathrm{End}(D)^*$ which extends α. Fix a metric d for the topology of X and define $\lambda : X \to [0,1]$ by $\lambda(x) = d(x, X \setminus$

$V)\big(d(x,X\setminus V)+d(x,Y)\big)^{-1}$. Let us define $\widehat{\alpha}:X\times[0,1]\to \mathrm{End}(D)$ by $\widehat{\alpha}_{(x,t)}=\alpha_V(x,\lambda(x)t)$ and observe that $\widehat{\alpha}$ extends α. Finally, since $\widehat{\alpha}_{(x,t)}$ is homotopic to $\widehat{\alpha}_{(x,0)}=\alpha_{(x,0)}$, we conclude that the image of $\widehat{\alpha}$ in contained in $\mathrm{End}(D)^*$. □

Proposition 6.4. *Let X be a compact metrizable space and let D be a KK-semiprojective stable Kirchberg algebra. Let A be a separable $C(X)$-algebra which is locally isomorphic to $C(X)\otimes D$. Suppose that there is $\sigma\in KK(D,A)$ such that $\sigma_x\in KK(D,A(x))^{-1}$ for all $x\in X$. Then there is an isomorphism of $C(X)$-algebras $\psi:C(X)\otimes D\to A$ such that $KK(\psi|_D)=\sigma$.*

Proof. Since X is compact and A is locally trivial it follows that $\mathrm{cat}_{\{D\}}(A)<\infty$. By Lemma 2.9, $A\cong pAp\otimes\mathcal{O}_\infty\otimes\mathcal{K}$ for some projection $p\in A$. By Theorem 3.1, there is a full $*$-homomorphism $\varphi:D\to A$ such that $KK(\varphi)=\sigma$. We shall construct an isomorphism of $C(X)$-algebras $\psi:C(X)\otimes D\to A$ such that ψ is homotopic to $\widetilde{\varphi}$, the $C(X)$-linear extension of φ. Moreover the homotopy $(H_t)_{t\in[0,1]}$ will have the property that $H_{(x,t)}:D\to A(x)$ is an isomorphism for all $x\in X$ and $t>0$. We prove this by induction on numbers n with the property that there are two closed covers of X, $W_1,...,W_n$ and $Y_1,...,Y_n$ such that Y_i contained in the interior of W_i and $A(W_i)\cong C(W_i)\otimes D$ for $1\le i\le n$. First we observe that the case $n=1$ follows from Proposition 6.2. Let us now pass from $n-1$ to n. Given two covers as above, there is yet another closed cover $V_1,...,V_n$ of X such that V_i is a neighborhood of Y_i and W_i is a neighborhood of V_i for all $1\le i\le n$. Set $Y=\cup_{i=1}^{n-1}Y_i$, $V=\cup_{i=1}^{n-1}V_i$ and $W=\cup_{i=1}^{n-1}W_i$. By the inductive hypothesis applied to $A(V)$, and the covers $V_1,...,V_{n-1}$ and $W_1\cap V,...,W_{n-1}\cap V$ there is a homotopy $h:D\to A(V)\otimes C[0,1]$ such that $h_{(x,0)}=\varphi_x$ and $h_{(x,t)}:D\to A(x)$ is an isomorphism for all $(x,t)\in V\times(0,1]$. Fix a trivialization $\nu:A(Y_{n+1})\to C(Y_{n+1})\otimes D$. Define a continuous map $\alpha:(V\cap Y_{n+1})\times[0,1]\cup Y_{n+1}\times\{0\}\to\mathrm{End}(D)$ by setting $\alpha_{(x,t)}=\nu_x h_{(x,t)}$ if $(x,t)\in(V\cap Y_{n+1})\times[0,1]$ and $\alpha_{(x,0)}=\nu_x\varphi_x$ if $x\in Y_{n+1}$. Since $V\cap Y_{n+1}$ is a neighborhood of $Y\cap Y_{n+1}$ in Y_{n+1} and since $\nu_x\varphi_x\in\mathrm{End}(D)^*$ for all $x\in Y_{n+1}$, by Lemma 6.3 there is a continuous map $\eta:Y_{n+1}\times[0,1]\to\mathrm{End}(D)^*$ which extends the restriction of α to $(Y\cap Y_{n+1})\times[0,1]\cup Y_{n+1}\times\{0\}$. We conclude the construction of the desired homotopy by defining $H:D\to A(X)\otimes C[0,1]$ by $H_{(x,t)}=h_{(x,t)}$ for $(x,t)\in Y\times[0,1]$ and $H_{(x,t)}=\nu_x^{-1}\eta_{(x,t)}$ for $(x,t)\in Y_{n+1}\times[0,1]$. □

Lemma 6.5. *Let D be a KK-semiprojective stable Kirchberg algebra. Let X be a compact metrizable space and Y, Z be closed subsets of X such that $X=Y\cup Z$. Suppose that $\gamma:D\to C(Y\cap Z)\otimes D$ is a full $*$-homomorphism which admits a lifting to a full $*$-homomorphism $\alpha:D\to C(Y)\otimes D$ such that $\alpha_x\in\mathrm{End}(D)^*$ for all $x\in Y$. Then the pullback $C(Y)\otimes D\oplus_{\pi_{Y\cap Z},\widetilde{\gamma}\pi_{Y\cap Z}}C(Z)\otimes D$ is isomorphic to $C(X)\otimes D$.*

Proof. By Prop. 6.2 there is a $*$-homomorphism $\eta:D\to C(Y)\otimes D$ such that $\eta_x=\gamma_x$ for $x\in Y\cap Z$ and such that $\eta_x\in\mathrm{Aut}(D)$ for $x\in Y\setminus Z$. Using the short five lemma one checks immediately that the triplet $(\widetilde{\eta},\widetilde{\gamma},\mathrm{id}_{C(Z)\otimes D})$ defines a $C(X)$-linear isomorphism:
$C(X)\otimes D=C(Y)\otimes D\oplus_{\pi_{Y\cap Z},\pi_{Y\cap Z}}C(Z)\otimes D\to C(Y)\otimes D\oplus_{\pi_{Y\cap Z},\widetilde{\gamma}\pi_{Y\cap Z}}C(Z)\otimes D$. □

Lemma 6.6. *Let D be a KK-semiprojective stable Kirchberg algebra. Let Y, Z and Z' be closed subsets of a compact metrizable space X such that Z' is a neighborhood of Z and $X=Y\cup Z$. Let B be a $C(Y)$-algebra locally isomorphic to $C(Y)\otimes D$ and let E be a $C(Z')$-algebra locally isomorphic to $C(Z')\otimes D$. Let $\alpha:E(Y\cap Z')\to B(Y\cap Z')$ be a $*$-monomorphism of $C(Y\cap Z')$-algebras such* *that* $KK(\alpha_x) \in KK(E(x), B(x))^{-1}$ *for all* $x \in Y \cap Z'$. *If* $\gamma = \alpha_{Y\cap Z}$, *then* $B(Y) \oplus_{\pi_{Y\cap Z},\gamma\,\pi_{Y\cap Z}} E(Z)$ *is locally isomorphic to* $C(X) \otimes D$.

Proof. Since we are dealing with a local property, we may assume that $B = C(Y) \otimes D$ and $E = C(Z') \otimes D$. To simplify notation we let π stand for both $\pi^Y_{Y\cap Z}$ and $\pi^Z_{Y\cap Z}$ in the sequel. Let us denote by H the $C(X)$-algebra $C(Y) \otimes D \oplus_{\pi,\gamma\pi} C(Z) \otimes D$. We must show that H is locally trivial. Let $x \in X$. If $x \notin Z$, then there is a closed neighborhood V of x which does not intersect Z, and hence the restriction of H to V is isomorphic to $C(V) \otimes D$, as it follows immediately from the definition of H. It remains to consider the case when $x \in Z$. Now Z' is a closed neighborhood of x in X and the restriction of H to Z' is isomorphic to $C(Y \cap Z') \otimes D \oplus_{\pi,\gamma\pi} C(Z) \otimes D$. Since $\gamma : Y \cap Z \to \mathrm{End}(D)^*$ admits a continuous extension $\alpha : Y \cap Z' \to \mathrm{End}(D)^*$, it follows that $H(Z')$ is isomorphic to $C(Z') \otimes D$ by Lemma 6.5. □

Proposition 6.7. *Let* X, A, D *and* σ *be as in Proposition 4.7. For any finite subset* $\mathcal{F}$ *of* A *and any* $\varepsilon > 0$ *there is a* $C(X)$*-algebra* B *which is locally isomorphic to* $C(X) \otimes D$ *and there exists a* $C(X)$*-linear* $*$*-monomorphism* $\eta : B \to A$ *such that* $\mathcal{F} \subset_\varepsilon \eta(B)$ *and* $KK(\eta_x) \in KK(B(x), A(x))^{-1}$ *for all* $x \in X$.

Proof. Let $\psi_k : E_k = C(Y_k) \otimes D \to A(Y_k)$, $k = 0, ..., n$ be as in the conclusion of Proposition 4.7, strengthen as in Remark 4.8. Therefore we may assume that there is another n-fibered $\{D\}$-monomorphism $(\psi'_0, ..., \psi'_n)$ into A such that $\psi'_k : C(Y'_k) \otimes D \to A(Y'_k)$, Y'_k is a closed neighborhood of Y_i, and $\pi_{Y_k}\psi'_k = \psi_k$, $k = 0, ..., n$. Let X_k, B_k, η_k and γ_k be as in Definition 2.8. B_0 and η_0 satisfy the conclusion of the proposition, except that we need to prove that B_0 is locally isomorphic to $C(X) \otimes D$. We prove by induction on decreasing k that the $C(X_k)$-algebras B_k are locally trivial. Indeed $B_n = C(X_n) \otimes D$ and assuming that B_k is locally trivial, it follows by Lemma 6.6 that B_{k-1} is locally trivial, since by (5)

$$B_{k-1} \cong B_k \oplus_{\pi\eta_k, \pi\psi_{k-1}} E_{k-1} \cong B_k \oplus_{\pi,\gamma_k\pi} E_{k-1}, \quad (\ \pi = \pi_{X_k \cap Y_{k-1}})$$

and $\gamma_k : E_{k-1}(X_k \cap Y_{k-1}) \to B_k(X_k \cap Y_{k-1})$, $(\gamma_k)_x = (\eta_k)_x^{-1}(\psi_{k-1})_x$, extends to a $*$-monomorphism $\alpha : E_{k-1}(X_k \cap Y'_{k-1}) \to B_k(X_k \cap Y'_{k-1})$, $\alpha_x = (\eta_k)_x^{-1}(\psi'_{k-1})_x$ and $KK(\alpha_x)$ is a KK-equivalence since both $KK((\eta_k)_x)$ and $KK((\psi_{k-1})_x)$ are KK-equivalences. □

7. When is a $C(X)$-algebra locally trivial

In this section we prove Theorems 1.1 – 1.5 and some of their consequences.

Proof of Theorem 1.2.

Proof. Let X denote the primitive spectrum of A. Then A is a continuous $C(X)$-algebra and its fibers are stable Kirchberg algebras (see [5, 2.2.2]). Since A is separable, X is metrizable by Lemma 2.2. By Proposition 6.7 there is a sequence of $C(X)$-algebras $(A_k)_{k=1}^\infty$ locally isomorphic to $C(X) \otimes D$ and a sequence of $C(X)$-linear $*$-monomorphisms $(\eta_k : A_k \to A)_{k=1}^\infty$, such that $KK(\eta_k)_x$ is a KK-equivalence for each $x \in X$ and $(\eta_k(A_k))_{k=1}^\infty$ is an exhaustive sequence of $C(X)$-subalgebras of A. Since D is weakly semiprojective and KK-stable, after passing to a subsequence of (A_k) if necessary, we find a sequence $(\sigma_k)_{k=1}^\infty$, $\sigma_k \in KK(D, A_k)$ such that $KK(\eta_k)\sigma_k = \sigma$ for all $k \geq 1$. Since both $KK(\eta_k)_x$ and σ_x are KK-equivalences, we deduce that $(\sigma_k)_x \in KK(D, A_k(x))^{-1}$ for all $x \in X$. By Proposition 6.4, for each $k \geq 1$ there is an isomorphism of $C(X)$-algebras

$\varphi_k : C(X)\otimes D \to A_k$ such that $KK(\varphi_k) = \sigma_k$. Therefore if we set $\theta_k = \eta_k\varphi_k$, then θ_k is a $C(X)$-linear $*$-monomorphism from $B \overset{def}{=} C(X)\otimes D$ to A such that $KK(\theta_k) = \sigma$ and $(\theta_k(B))_{k=1}^{\infty}$ is an exhaustive sequence of $C(X)$-subalgebras of A. Using again the weak semiprojectivity and the KK-stability of D, and Lemma 4.1, after passing to a subsequence of $(\theta_k)_{k=1}^{\infty}$, we construct a sequence of finite sets $\mathcal{F}_k \subset B$ and a sequence of $C(X)$-linear $*$-monomorphisms $\mu_k : B \to B$ such that

(i) $KK(\theta_{k+1}\mu_k) = KK(\theta_k)$ for all $k \geq 1$,
(ii) $\|\theta_{k+1}\mu_k(a) - \theta_k(a)\| < 2^{-k}$ for all $a \in \mathcal{F}_k$ and all $k \geq 1$,
(iii) $\mu_k(\mathcal{F}_k) \subset \mathcal{F}_{k+1}$ for all $k \geq 1$,
(iv) $\bigcup_{j=k+1}^{\infty} \big(\mu_{j-1}\circ\cdots\circ\mu_k\big)^{-1}(\mathcal{F}_j)$ is dense in B and $\bigcup_{j=k}^{\infty} \theta_j(\mathcal{F}_j)$ is dense in A for all $k \geq 1$.

Arguing as in the proof of [29, Prop. 2.3.2], one verifies that

$$\Delta_k(a) = \lim_{j\to\infty} \theta_j \circ \big(\mu_{j-1}\circ\cdots\circ\mu_k\big)(a)$$

defines a sequence of $*$-monomorphisms $\Delta_k : B \to A$ such that $\Delta_{k+1}\mu_k = \Delta_k$ and the induced map $\Delta : \varinjlim_k (B,\mu_k) \to A$ is an isomorphism of $C(X)$-algebras. Let us show that $\varinjlim_k (B,\mu_k)$ is isomorphic to B. To this purpose, in view of Elliott's intertwining argument, it suffices to show that each map μ_k is approximately unitarily equivalent to a $C(X)$-linear automorphism of B. Since $KK(\theta_k) = \sigma$, we deduce from (i) that $KK((\mu_k)_x) = KK(\mathrm{id}_D)$ for all $x \in X$. By Proposition 6.1, this property implies that each map μ_k is approximately unitarily equivalent to a $C(X)$-linear automorphism of B. Therefore there is an isomorphism of $C(X)$-algebras $\Delta : B \to A$. Let us show that we can arrange that $KK(\Delta|_D) = \sigma$. By Theorem 3.1, there is a full $*$-homomorphism $\alpha : D \to B$ such that $KK(\alpha) = KK(\Delta^{-1})\sigma$. Since $KK(\Delta_x^{-1})\sigma_x \in KK(D,D)^{-1}$, by Proposition 6.1 there is $\Phi_1 : D \to C(X)\otimes D$ such that $\widetilde{\Phi}_1 \in \mathrm{Aut}_{C(X)}(B)$ and $KK(\Phi_1) = KK(\Delta^{-1})\sigma$. Then $\Phi = \Delta\widetilde{\Phi}_1 : B \to A$ is an isomorphism such that $KK(\Phi|_D) = KK(\Delta\Phi_1) = \sigma$. □

Proposition 7.1. *Let ψ be a full endomorphism of a Kirchberg algebra. If D is unital we assume that $\psi(1) = 1$ as well. Then the continuous $C[0,1]$-algebra $E = \{f \in C[0,1]\otimes D : f(0) \in \psi(D)\}$ is locally trivial if and only if ψ is homotopic to an automorphism of D.*

Proof. Suppose that E is trivial on some neighborhood of 0. Thus there is $s \in (0,1]$ such that $C[0,s]\otimes D \cong E[0,s]$. Since $E[0,s] \subset C[0,s]\otimes D$, there is a continuous path $(\theta_t)_{t\in[0,s]}$ in $\mathrm{End}(D)$ such that $\theta_t \in \mathrm{Aut}(D)$ for $0 < t \leq s$ and $\theta_0(D) = \psi(D)$. Set $\beta = \theta_0^{-1}\psi \in \mathrm{Aut}(D)$. Then ψ is homotopic to an automorphism via the path $(\theta_t\beta)_{t\in[0,s]}$. Conversely, if ψ is homotopic to an automorphism α, then by Theorem 3.1 there is a continuous path $(u_t)_{t\in(0,1]}$ of unitaries in D^+ such that $\lim_{t\to 0}\|\psi(d) - u_t\alpha(d)u_t^*\| = 0$ for all $d \in D$. The path $(\theta_t)_{t\in[0,1]}$ defined by $\theta_0 = \psi$ and $\theta_t = u_t\alpha u_t^*$ for $t \in (0,1]$ induces a $C[0,1]$-linear $*$-endomorphism of $C[0,1]\otimes D$ which maps injectively $C[0,1]\otimes D$ onto E.

□

Proof of Theorem 1.3.

Proof. For the first part we apply Theorem 1.2 for $D = \mathcal{O}_2\otimes\mathcal{K}$ and $\sigma = 0$. For the second part we assert that if D is a Kirchberg such that all continuous $C[0,1]$-algebras with fibers isomorphic to D are locally trivial then D is stable and $KK(D,D) = 0$. This implies that D is KK-equivalent to $\mathcal{O}_2$ and hence that $D \cong \mathcal{O}_2 \otimes \mathcal{K}$ by [29, Thm. 8.4.1]. The Kirchberg algebra D is either unital or stable [29, Prop. 4.1.3]. Let $\psi : D \to D$ be a $*$-monomorphism such that $KK(\psi) = 0$ and such that $\psi(1_D) < 1_D$ if D is unital. By Proposition 7.1 ψ is homotopic to an automorphism of θ of D. Therefore D must be nonunital (and hence stable), since otherwise 1_D would be homotopic to its proper subprojection $\psi(1_D)$. Moreover $KK(\theta) = KK(\psi) = 0$ and hence $KK(D, D) = 0$ since θ is an automorphism. $\square$

Dixmier and Douady [12] proved that a continuous field with fibers $\mathcal{K}$ over a finite dimensional locally compact Hausdorff space is locally trivial if and only it verifies Fell's condition, i.e. for each $x_0 \in X$ there is a continuous section a of the field such that $a(x)$ is a rank one projection for each x in a neighborhood of x_0. We have a analogous result:

Corollary 7.2. *Let A be a separable C*-algebra whose primitive spectrum X is Hausdorff and of finite dimension. Suppose that for each $x \in X$, $A(x)$ is KK-semiprojective, nuclear, purely infinite and stable. Then A is locally trivial if and only if for each $x \in X$ there exist a closed neighborhood V of x, a Kirchberg algebra D and $\sigma \in KK(D, A(V))$ such that $\sigma_v \in KK(D, A(v))^{-1}$ for each $v \in V$.*

Proof. One applies Theorem 1.2 for $D \otimes \mathcal{K}$ and $A(V)$. $\square$

We turn now to unital $C(X)$-algebras.

Theorem 7.3. *Let A be a separable unital $C(X)$-algebra over a finite dimensional compact Hausdorff space X. Suppose that each fiber $A(x)$ is nuclear simple and purely infinite. Then A is isomorphic to $C(X) \otimes D$, for some KK-semiprojective unital Kirchberg algebra D, if and only if there is $\sigma \in KK(D, A)$ such that $K_0(\sigma)[1_D] = [1_A]$ and $\sigma_x \in KK(D, A(x))^{-1}$ for all $x \in X$. For any such σ there is an isomorphism of $C(X)$-algebras $\Phi : C(X) \otimes D \to A$ such that $KK(\Phi|_D) = \sigma$.*

Proof. We verify the nontrivial implication. X is metrizable by Lemma 2.2. A is a continuous $C(X)$-algebra by Lemma 2.3. By Theorem 1.2, there is an isomorphism $\Phi : C(X) \otimes D \otimes \mathcal{K} \to A \otimes \mathcal{K}$ such that $KK(\Phi) = \sigma$. Since $K_0(\sigma)[1_D] = [1_A]$, and since $A \otimes \mathcal{K}$ contains a full properly infinite projection, we may arrange that $\Phi(1_{C(X)\otimes D} \otimes e_{11}) = 1_A \otimes e_{11}$ after conjugating Φ by some unitary $u \in M(A \otimes \mathcal{K})$. Then $\varphi = \Phi|_{C(X)\otimes D \otimes e_{11}}$ satisfies the conclusion of the theorem. $\square$

Proof of Theorem 1.4.

Proof. Let D be a KK-semiprojective unital Kirchberg algebra D such that every unital $*$-endomorphism of D is a KK-equivalence. Suppose that A is a separable unital $C(X)$-algebra over a finite dimensional compact Hausdorff space the fibers of which are isomorphic to D. We shall prove that A is locally trivial. By Theorem 7.3, it suffices to show that each point $x_0 \in X$ has a closed neighborhood V for which there is $\sigma \in KK(D, A(V))$ such that $K_0(\sigma)[1_D] = [1_{A(V)}]$ and $\sigma_x \in KK(D, A(x))^{-1}$ for all $x \in V$.

Let $(V_n)_{n=1}^{\infty}$ be a decreasing sequence of closed neighborhoods of x_0 whose intersection is $\{x_0\}$. Then $A(x_0) \cong \varinjlim A(V_n)$. By assumption, there is an isomorphism $\eta : D \to A(x_0)$. Since D is KK-semiprojective, there is $m \geq 1$ such that $KK(\eta)$ lifts to some $\sigma \in KK(D, A(V_m))$ such that $K_0(\sigma)[1_D] = [1_{A(V_m)}]$. Let $x \in V_m$. By assumption, there is an isomorphism $\varphi : A(x) \to D$. The K_0-morphism induced by $KK(\varphi)\sigma_x$ maps $[1_D]$ to itself. By Theorem 3.1 there is a unital

$*$-homomorphism $\psi : D \to D$ such that $KK(\psi) = KK(\varphi)\sigma_x$. By assumption we must have $KK(\psi) \in KK(D,D)^{-1}$ and hence $\sigma_x \in KK(D, A(x))^{-1}$ since φ is an isomorphism. Therefore $A(V_m) \cong C(V_m) \otimes D$ by Theorem 7.3.

Conversely, let us assume that all separable unital continuous $C[0,1]$-algebras with fibers isomorphic to D are locally trivial. Let ψ be any unital $*$-endomorphism of D. By Proposition 7.1 ψ is homotopic to an automorphism of D and hence $KK(\psi)$ is invertible. $\square$

Proof of Theorem 1.1

Proof. Let A be as in Theorem 1.1 and let $n \in \{2, 3, ...\} \cup \{\infty\}$. It is known that $\mathcal{O}_n$ satisfies the UCT. Moreover $K_0(\mathcal{O}_n)$ is generated by $[1_{\mathcal{O}_n}]$ and $K_1(\mathcal{O}_n) = 0$. Therefore any unital $*$-endomorphism of $\mathcal{O}_n$ is a KK-equivalence. It follows that A is locally trivial by Theorem 1.4. Suppose now that $n = 2$. Since $KK(\mathcal{O}_2, \mathcal{O}_2) = KK(\mathcal{O}_2, A) = 0$, we may apply Theorem 1.4 with $\sigma = 0$ and obtain that $A \cong C(X) \otimes \mathcal{O}_2$. Suppose now that $n = \infty$. Let us define $\theta : K_0(\mathcal{O}_\infty) \to K_0(A)$ by $\theta(k[1_{\mathcal{O}_\infty}]) = k[1_A]$, $k \in \mathbb{Z}$. Since $\mathcal{O}_\infty$ satisfies the UCT, θ lifts to some element $\sigma \in KK(\mathcal{O}_\infty, A)$. By Theorem 1.4 it follows that $A \cong C(X) \otimes \mathcal{O}_\infty$. Finally let us consider the case $n \in \{3, 4, ...\}$. Then $K_0(\mathcal{O}_n) = \mathbb{Z}/(n-1)$. Since $\mathcal{O}_n$ satisfies the UCT, the existence of an element $\sigma \in KK(\mathcal{O}_n, A)$ such that $K_0(\sigma)[1_{\mathcal{O}_n}] = [1_A]$ is equivalent to the existence of a morphism of groups $\theta : \mathbb{Z}/(n-1) \to K_0(A)$ such that $\theta(\bar{1}) = [1_A]$. This is equivalent to requiring that $(n-1)[1_A] = 0$. $\square$

As a corollary of Theorem 1.1 we have that $[X, \mathrm{Aut}(\mathcal{O}_\infty)]$ reduces to a point. The homotopy groups of the endomorphisms of the stable Cuntz-Krieger algebras were computed in [7]. Let $v_1, \dots, v_n$ be the canonical generators of $\mathcal{O}_n$, $2 \le n < \infty$.

Theorem 7.4. *For any compact metrizable space X there is a bijection $[X, \mathrm{Aut}(\mathcal{O}_n)] \to K_1(C(X) \otimes \mathcal{O}_n)$. The k^{th}-homotopy group $\pi_k(\mathrm{Aut}(\mathcal{O}_n))$ is isomorphic to $\mathbb{Z}/(n-1)$ if k is odd and it vanishes if k is even. In particular $\pi_1(\mathrm{Aut}(\mathcal{O}_n))$ is generated by the class of the canonical action of $\mathbb{T}$ on $\mathcal{O}_n$, $\lambda_z(v_i) = zv_i$.*

Proof. Since $\mathcal{O}_n$ satisfies the UCT, we deduce that $\mathrm{End}(\mathcal{O}_n)^* = \mathrm{End}(\mathcal{O}_n)$. An immediate application of Proposition 6.1 shows that the natural map $\mathrm{Aut}(\mathcal{O}_n) \hookrightarrow \mathrm{End}(\mathcal{O}_n)$ induces an isomorphism of groups $[X, \mathrm{Aut}(\mathcal{O}_n)] \cong [X, \mathrm{End}(\mathcal{O}_n)]$. Let $\iota : \mathcal{O}_n \to C(X) \otimes \mathcal{O}_n$ be defined by $\iota(v_i) = 1_{C(X)} \otimes v_i$, $i = 1, ..., n$. The map $\psi \mapsto u(\psi) = \psi(v_1)\iota(v_1)^* + \cdots + \psi(v_n)\iota(v_n)^*$ is known to be a homeomorphism from $\mathrm{Hom}(\mathcal{O}_n, C(X) \otimes \mathcal{O}_n)$ to the unitary group of $C(X) \otimes \mathcal{O}_n$. Its inverse maps a unitary w to the $*$-homomorphism ψ uniquely defined by $\psi(v_i) = w\iota(v_i)$, $i = 1, ..., n$. Therefore

$$[X, \mathrm{Aut}(\mathcal{O}_n)] \cong [X, \mathrm{End}(\mathcal{O}_n)] \cong \pi_0(U(C(X) \otimes \mathcal{O}_n)) \cong K_1(C(X) \otimes \mathcal{O}_n).$$

The last isomorphism holds since $\pi_0(U(B)) \cong K_1(B)$ if $B \cong B \otimes \mathcal{O}_\infty$, by [28, Lemma 2.1.7]. One verifies easily that if $\varphi \in \mathrm{Hom}(\mathcal{O}_n, C(X) \otimes \mathcal{O}_n)$, then $u(\widetilde{\psi}\varphi) = \widetilde{\psi}(u(\varphi))u(\psi)$. Therefore the bijection $\chi : [X, \mathrm{End}(D)] \to K_1(C(X) \otimes \mathcal{O}_n)$ is an isomorphism of groups whenever $K_1(\widetilde{\psi}) = \mathrm{id}$ for all $\psi \in \mathrm{Hom}(\mathcal{O}_n, C(X) \otimes \mathcal{O}_n)$. Using the $C(X)$-linearity of $\widetilde{\psi}$ one observes that this holds if the $n-1$ torsion of $K_0(C(X))$ reduces to $\{0\}$, since in that case the map $K_1(C(X)) \to K_1(C(X) \otimes \mathcal{O}_n)$ is surjective by the Künneth formula. $\square$

Corollary 7.5. *Let X be a finite dimensional compact metrizable space. The isomorphism classes of unital separable $C(SX)$-algebras with all fibers isomorphic to $\mathcal{O}_n$ are parameterized by $K_1(C(X)\otimes \mathcal{O}_n)$.*

Proof. This follows from Theorems 1.1 and 7.4, since the locally trivial principal H-bundles over $SX = X \times [0,1]/X \times \{0,1\}$ are parameterized by the homotopy classes $[X, H]$ if H is a path connected group [17, Cor. 8.4]. Here we take $H = \mathrm{Aut}(\mathcal{O}_n)$. □

Examples of nontrivial unital $C(X)$-algebras with fiber $\mathcal{O}_n$ over a $2m$-sphere arising from vector bundles were exhibited in [36], see also [35].

We need some preparation for the proof of Theorem 1.5. Let G be a group, let $g \in G$ and set $\mathrm{End}(G, g) = \{\alpha \in \mathrm{End}(G) : \alpha(g) = g\}$. The pair (G, g) is called *weakly rigid* if $\mathrm{End}(G, g) \subset \mathrm{Aut}(G)$ and *rigid* if $\mathrm{End}(G, g) = \{\mathrm{id}_G\}$.

Theorem 7.6. *If G is a finitely generated abelian group, then (G, g) is* weakly rigid *if and only if (G, g) is isomorphic to one of the pointed groups from the list $\mathcal{G}$ of Theorem 1.5.*

Proof. First we make a number of remarks.

(1) (G, g) is weakly rigid if and only if $(G, \alpha(g))$ is weakly rigid for some (or any) $\alpha \in \mathrm{Aut}(G)$. Indeed if $\beta \in \mathrm{End}(G, g)$ then $\alpha\beta\alpha^{-1} \in \mathrm{End}(G, \alpha(g))$.

(2) By considering the zero endomorphism of G we see that if (G, g) is weakly rigid and $G \neq 0$ then $g \neq 0$.

(3) If $(G \oplus H, g \oplus h)$ is weakly rigid, then so are (G, g) and (H, h).

(4) Let us observe that $(\mathbb{Z}^2, g)$ is not weakly rigid for any g. Indeed, if $g = (a, b) \neq 0$, then the matrix $\begin{pmatrix} 1+b^2 & -ab \\ -ab & 1+a^2 \end{pmatrix}$ defines an endomorphism α of $\mathbb{Z}^2$ such that $\alpha(g) = g$, but α is not invertible since $\det(\alpha) = 1 + a^2 + b^2 > 1$.

(5) Let p be a prime and let $1 \leq e_1 \leq e_2$, $0 \leq s_1 < e_1$, $0 \leq s_2 < e_2$ be integers. If $(G, g) = (\mathbb{Z}/p^{e_1} \oplus \mathbb{Z}/p^{e_2}, p^{s_1} \oplus p^{s_2})$ is weakly rigid then $0 < s_2 - s_1 < e_2 - e_1$. Indeed if $s_1 \geq s_2$ then the matrix $\begin{pmatrix} 0 & p^{s_1-s_2} \\ 0 & 1 \end{pmatrix}$ induces a noninjective endomorphism of (G, g). Also if $s_1 < s_2$ and $s_2 - s_1 \geq e_2 - e_1$ then $p^{e_1}\bar{b} = 0$ in $\mathbb{Z}/p^{e_2}$, where $b = p^{s_2-s_1}$ and so the matrix $\begin{pmatrix} 1 & 0 \\ b & 0 \end{pmatrix}$ induces a well-defined noninjective endomorphism of (G, g).

(6) Let p be a prime and let $1 \leq k$, $0 \leq s < e$ be integers. Suppose that $(\mathbb{Z} \oplus \mathbb{Z}/p^e, k \oplus p^s)$ is weakly rigid. Then k is divisible by p^{s+1}. Indeed, seeking a contradiction suppose that k can be written as $k = p^t c$ where $0 \leq t \leq s$ and c are integers such that c is not divisible by p. Let d be an integer such that $dc - 1$ is divisible by p^e. Then the matrix $\begin{pmatrix} 1 & 0 \\ dp^{s-t} & 0 \end{pmatrix}$ induces a noninjective endomorphism of $(\mathbb{Z} \oplus \mathbb{Z}/p^e, k \oplus p^s)$.

Suppose now that (G, g) is weakly rigid. We shall show that (G, g) is isomorphic to one of the pointed groups from the list $\mathcal{G}$. Since G is abelian and finitely generated it decomposes as a direct sum of its primary components

$$G \cong \mathbb{Z}^r \oplus G(p_1) \oplus \cdots \oplus G(p_m) \tag{20}$$

where p_i are distinct prime numbers. Each primary component $G(p_i)$ is of the form

$$G(p_i) = \mathbb{Z}/p_i^{e_{i\,1}} \oplus \cdots \oplus Z/p_i^{e_{i\,n(i)}} \tag{21}$$

where $1 \le e_{i\,1} \le \cdots \le e_{i\,n(i)}$ are positive integers. Corresponding to the decomposition (20) we write the base point $g = g_0 \oplus g_1 \oplus ... \oplus g_m$ with $g_0 \in \mathbb{Z}^r$ and $g_i \in G(p_i)$ for $i \ge 1$. If g_{ij} is the component of g_i in $\mathbb{Z}/p^{e_{ij}}$, then it follows from (1), (2) and (3) that we may assume that $g_{ij} = p^{s_{ij}}$ for some integer $0 \le s_{ij} < e_{ij}$. Using (3) and (4) we deduce that $r = 1$ in (20) and that $g_0 = k \ne 0$ by (2). We may assume that $k \ge 1$ by (1). Then using (3) and (5) we deduce that for each $1 \le i \le m$, $0 < s_{i\,j+1} - s_{i\,j} < e_{i\,j+1} - e_{i\,j}$ for $1 \le j < n(i)$. Finally, from (3) and (6) we see that k is divisible by the product $p_1^{s_{1\,n(1)}} \cdots p_m^{s_{m\,n(m)}}$. Therefore (G, g) is isomorphic to one of the pointed groups on the list $\mathcal{G}$.

Conversely, we shall prove that if (G, g) belongs to the list $\mathcal{G}$ then (G, g) is weakly rigid. This is obvious if G is torsion free i.e for $(\{0\}, 0)$ and $(\mathbb{Z}, k)$ with $k \ge 1$.

Let us consider the case when G is a torsion group. Since

$$\mathrm{End}\big(G(p_1) \oplus \cdots \oplus G(p_m), g_1 \oplus \cdots \oplus g_m\big) \cong \bigoplus_{i=1}^{m} \mathrm{End}\big(G(p_i), g_i\big)$$

it suffices to assume that G is a p-group,

$$(G, g) = \big(\mathbb{Z}/p^{e_1} \oplus \cdots \oplus Z/p^{e_n}, p^{s_1} \oplus \cdots \oplus p^{s_n}\big)$$

with $0 \le s_i < e_i$ for $i = 1, ..., n$ and $0 < s_{i+1} - s_i < e_{i+1} - e_i$ for $1 \le i < n$. For each $0 \le i, j \le n$ set $e_{ij} = \max\{e_i - e_j, 0\}$. It follows immediately that $s_i < e_{ij} + s_j$ for all $i \ne j$. Let $\alpha \in \mathrm{End}(G, g)$. It is well-known that α is induced by a square matrix $A = [a_{ij}] \in M_n(\mathbb{Z})$ with the property that each entry a_{ij} is divisible by $p^{e_{ij}}$ and so $a_{ij} = p^{e_{ij}} b_{ij}$ for some $b_{ij} \in Z$, see [16]. Since $\alpha(g) = g$, we have $\sum_{j=1}^{n} \bar{b}_{ij} p^{e_{ij}+s_j} = p^{s_i}$ in $\mathbb{Z}/p^{e_i}$ for all $0 \le i \le n$. Since $e_{ij} + s_j > s_i$ for $i \ne j$ and $e_i > s_i$ we see that $b_{ii} - 1$ must be divisible by p for all $1 \le i \le n$. Since $\det(A)$ is congruent to $b_{11} \cdots b_{nn}$ modulo p it follows that $\det(A)$ is not divisible by p and so $\alpha \in \mathrm{Aut}(G)$ by [16].

Finally consider the case when $(G, g) = \big(\mathbb{Z} \oplus G(p_1) \oplus \cdots \oplus G(p_m), k \oplus g_1 \oplus \cdots \oplus g_m\big)$. If $\gamma \in \mathrm{End}(G, g)$ then there exist $\alpha_i \in \mathrm{End}(G(p_i), g_i)$ and $d_i \in G(p_i)$, $1 \le i \le n$, such that $\gamma(x_0 \oplus x_1 \oplus ... \oplus x_m) = x_0 \oplus (\alpha_1(x_1) + x_0 d_1) \oplus ... \oplus (\alpha_m(x_1) + x_0 d_m)$. Note that if each α_i is an automorphism then so is γ. Indeed, its inverse is $\gamma^{-1}(x_0 \oplus x_1 \oplus ... \oplus x_m) = x_0 \oplus (\alpha_1^{-1}(x_1) + x_0 c_1) \oplus ... \oplus (\alpha_m(x_1)^{-1} + x_0 c_m)$, where $c_i = -\alpha_1^{-1}(d_i)$. Therefore it suffices to consider the case $m = 1$, i.e.

$$(G, g) = \big(\mathbb{Z} \oplus \mathbb{Z}/p^{e_1} \oplus \cdots \oplus Z/p^{e_n}, k \oplus p^{s_1} \oplus \cdots \oplus p^{s_n}\big),$$

and (G, g) is on the list $\mathcal{G}$ (e). In particular $k = p^{s_n+1}\ell$ for some $\ell \in \mathbb{Z}$. Let $\gamma \in \mathrm{End}(G, g)$. Then there exists $\alpha \in \mathrm{End}(G(p))$ and $d \in G(p)$ such that $\gamma(x_0 \oplus x) = x_0 \oplus (\alpha(x) + x_0 d)$. Just as above, α is induced by a square matrix $A \in M_n(\mathbb{Z})$ of the form $A = [b_{ij} p^{e_{ij}}] \in M_n(\mathbb{Z})$ with $b_{ij} \in \mathbb{Z}$, $e_{ij} = \max\{e_i - e_j, 0\}$ and $d \in \mathbb{Z}^n$. Since $\gamma(g) = g$ we have that $p^{s_n+1}\ell d_i + \sum_{j=1}^{n} \bar{b}_{ij} p^{e_{ij}+s_j} = p^{s_i}$ in $\mathbb{Z}/p^{e_i}$ for all $0 \le i \le n$, where the d_i are the components of d. By reasoning as in the case when G was a torsion group considered above, since $s_n + 1 > s_i$ for all $1 \le i \le n$, $e_{ij} + s_j > s_i$ for all $i \ne j$ and $e_i > s_i$, it follows again that each $b_{ii} - 1$ is divisible by p and that the endomorphism α of $G(p)$ induced by the matrix A is an automorphism. We conclude that γ is an automorphism. $\square$

Proof of Theorem 1.5

Proof. (ii) and (iii) Let D be a unital Kirchberg algebra such that D satisfies the UCT and $K_*(D)$ is finitely generated. Then D is KK-semiprojective by Proposition 3.14 and $KK(D,D)^{-1} = \{\alpha \in KK(D,D) : K_*(\alpha) \text{ is bijective}\}$. By Theorem 3.1, all unital endomorphisms of D are KK-equivalences if and only if $(K_0(D) \oplus K_1(D), [1_D] \oplus 0)$ is weakly rigid. Equivalently, $K_1(D) = 0$ and $(K_0(D), [1_D])$ is weakly rigid. By Theorem 7.6 $(K_0(D), [1_D])$ is weakly rigid if and only if it isomorphic to one pointed groups from the list $\mathcal{G}$ of Theorem 1.5. We conclude the proof of (ii) and (iii) by applying Theorem 1.4.

(i) By Theorem 1.1 both $\mathcal{O}_2$ and $\mathcal{O}_\infty$ have the automatic triviality property. Conversely, let D be a unital Kirchberg algebra such that D satisfies the UCT and $K_*(D)$ is finitely generated and suppose that D has the automatic triviality property. We shall prove that D is isomorphic to either $\mathcal{O}_2$ or $\mathcal{O}_\infty$.

Let Y be a finite connected CW-complex and let $\iota : D \to C(Y) \otimes D$ be the map $\iota(d) = 1 \otimes d$. Let $[D, C(Y) \otimes D]$ denote the homotopy classes of unital $*$-homomorphisms from D to $C(Y) \otimes D$. By Theorem 3.1 the image of the map $\Delta : [D, C(Y) \otimes D] \to KK(D, C(Y) \otimes D)$ defined by $[\varphi] \mapsto KK(\varphi) - KK(\iota)$ coincides with the kernel of the restriction morphism $\rho : KK(D, C(Y) \otimes D) \to KK(\mathbb{C}1_D, C(Y) \otimes D)$.

We claim that $\ker \rho$ must vanish for all Y. Let $h \in \ker \rho$. Then there is a unital $*$-homomorphism $\varphi : D \to C(Y) \otimes D$ such that $\Delta[\varphi] = h$. By Theorem 1.4, each unital endomorphism of D induces a KK-equivalence. Therefore, by Proposition 6.1 there is a $*$-homomorphism $\Phi : D \to C(Y) \otimes D$ such that $\Phi_y \in \mathrm{Aut}(D)$ for all $y \in Y$ and $KK(\Phi) = KK(\varphi)$. Therefore $\Delta[\Phi] = KK(\Phi) - KK(\iota) = h$. By hypothesis, the $\mathrm{Aut}(D)$-principal bundle constructed over the suspension of Y with characteristic map $y \mapsto \Phi_y$ is trivial. It follows then from [17, Thm. 8.2 p85] that this map is homotopic to the to the constant map $Y \to \mathrm{Aut}(D)$ which shrinks Y to id_D. This implies that Φ is homotopic to ι and hence $h = 0$.

Let us now observe that $\ker \rho$ contains subgroups isomorphic to $\mathrm{Hom}(K_1(D), K_1(D))$ and $\mathrm{Ext}(K_0(D), K_0(D))$ if $Y = \mathbb{T}$, since D satisfies the UCT. It follows that both these groups must vanish and so $K_1(D) = 0$ and $K_0(D)$ is torsion free. On the other hand, $(K_0(D), [1_D])$ is weakly rigid by the first part of the proof. Since $K_0(D)$ is torsion free we deduce from Theorem 7.6 that either $K_0(D) = 0$ in which case $D \cong \mathcal{O}_2$ or that $(K_0(D), [1_D]) \cong (\mathbb{Z}, k)$, $k \geq 1$, in which case $D \cong M_k(\mathcal{O}_\infty)$ by the classification theorem of Kirchberg and Phillips.

To conclude the proof, it suffices to show that $\ker \rho \neq 0$ if $D = M_k(\mathcal{O}_\infty)$, $k \geq 2$ and Y is the two-dimensional space obtained by attaching a disk to a circle by a degree-k map. Since $K_0(C(Y) \otimes \mathcal{O}_\infty) \cong \mathbb{Z} \oplus \mathbb{Z}/k$ we can identify the map ρ with the map $\mathbb{Z} \oplus \mathbb{Z}/k \to \mathbb{Z} \oplus \mathbb{Z}/k$, $x \mapsto kx$ and so $\ker \rho \cong \mathbb{Z}/k$. □

Added in proof. Some of the results from this paper are further developed in [9]. Theorem 1.2 was shown to hold for all stable Kirchberg algebras D. The assumption that X is finite dimensional is essential Theorem 1.1. Theorem 1.5 (ii) extends as follows: $\mathcal{O}_2$, $\mathcal{O}_\infty$ and $B \otimes \mathcal{O}_\infty$ where B is a unital UHF algebras of infinite type are the only unital Kirchberg algebras which satisfy the UCT and have the automatic triviality property.

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