CONTINUOUS FIELDS OF C*-ALGEBRAS OVER FINITE DIMENSIONAL SPACES

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Abstract. Let $X$ be a finite dimensional compact metrizable space. We study a technique which employs semiprojectivity as a tool to produce approximations of $C(X)$-algebras by $C(X)$-subalgebras with controlled complexity. The following applications are given. All unital separable continuous fields of $C^*$-algebras over $X$ with fibers isomorphic to a fixed Cuntz algebra $O_n$, $n \in \{2, 3, \ldots, \infty\}$ are locally trivial. They are trivial if $n = 2$ or $n = \infty$. For $n \geq 3$ finite, such a field is trivial if and only if $(n - 1)[1_A] = 0$ in $K_0(A)$, where $A$ is the $C^*$-algebra of continuous sections of the field. We give a complete list of the Kirchberg algebras $D$ satisfying the UCT and having finitely generated $K$-theory groups for which every unital separable continuous field over $X$ with fibers isomorphic to $D$ is automatically locally trivial or trivial. In a more general context, we show that a separable unital continuous field over $X$ with fibers isomorphic to a $KK$-semiprojective Kirchberg $C^*$-algebra is trivial if and only if it satisfies a $K$-theoretical Fell type condition.

Contents

1. Introduction 1
2. $C(X)$-algebras 4
3. Semiprojectivity 8
4. Approximation of $C(X)$-algebras 13
5. Representing $C(X)$-algebras as inductive limits 18
6. When is a fibered product locally trivial 20
7. When is a $C(X)$-algebra locally trivial 23
References 29

1. Introduction

Gelfand’s characterization of commutative $C^*$-algebras has suggested the problem of representing non-commutative $C^*$-algebras as sections of bundles. By a result of Fell [15], if the primitive spectrum $X$ of a separable $C^*$-algebra $A$ is Hausdorff, then $A$ is isomorphic to the $C^*$-algebra of continuous sections vanishing at infinity of a continuous field of simple $C^*$-algebras over $X$. In particular $A$ is a continuous $C(X)$-algebra in the sense of Kasparov [18]. This description is very

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CONTINUOUS FIELDS OF $\text{C}^*$-ALGEBRAS OVER FINITE DIMENSIONAL SPACES

satisfactory, since as explained in [4], the continuous fields of $\text{C}^*$-algebras are in natural correspondence with the bundles of $\text{C}^*$-algebras in the sense of topology. Nevertheless, only a tiny fraction of the continuous fields of $\text{C}^*$-algebras correspond to locally trivial bundles.

In this paper we prove automatic and conditional local/global trivialization results for continuous fields of Kirchberg algebras. By a Kirchberg algebra we mean a purely infinite simple nuclear separable $\text{C}^*$-algebra [29]. Notable examples include the simple Cuntz-Krieger algebras [8]. The following theorem illustrates our results.

**Theorem 1.1.** A separable unital $\text{C}(X)$-algebra $A$ over a finite dimensional compact Hausdorff space $X$ all of whose fibers are isomorphic to the same Cuntz algebra $\mathcal{O}_n$, $n \in \{2, 3, \ldots, \infty\}$, is locally trivial. If $n = 2$ or $n = \infty$, then $A \cong \text{C}(X) \otimes \mathcal{O}_n$. If $3 \leq n < \infty$, then $A$ is isomorphic to $\text{C}(X) \otimes \mathcal{O}_n$ if and only if $(n - 1)[1_A] = 0$ in $K_0(A)$.

The case $X = [0, 1]$ of Theorem 1.1 was proved in a joint paper with G. Elliott [10].

We parametrize the homotopy classes $[X, \text{Aut}(\mathcal{O}_n)] \cong \begin{cases} K_1(\text{C}(X) \otimes \mathcal{O}_n) & \text{if } 3 \leq n < \infty, \\ \{\ast\} & \text{if } n = 2, \infty, \end{cases}$ (see Theorem 7.4) and hence classify the unital separable $\text{C}(SX)$-algebras $A$ with fiber $\mathcal{O}_n$ over the suspension $SX$ of a finite dimensional metrizable Hausdorff space $X$.

To put our results in perspective, let us recall that none of the general basic properties of a continuous field implies any kind of local triviality. An example of a continuous field of Kirchberg algebras over $[0, 1]$ which is not locally trivial at any point even though all of its fibers are mutually isomorphic is exhibited in [10, Ex. 8.4]. Examples of nonexact continuous fields with similar properties were found by S. Wassermann [37].

A separable $\text{C}^*$-algebra $D$ is $\text{KK}$-semiprojective if the functor $\text{KK}(D, -)$ is continuous, see Sec. 3. The class of $\text{KK}$-semiprojective $\text{C}^*$-algebras includes the nuclear semiprojective $\text{C}^*$-algebras and also the $\text{C}^*$-algebras which satisfy the Universal Coefficient Theorem in $\text{KK}$-theory (abbreviated UCT [31]) and whose K-theory groups are finitely generated. It is very interesting that the only obstruction to local or global triviality for a continuous field of Kirchberg algebras is of purely $\text{K}$-theoretical nature.

**Theorem 1.2.** Let $A$ be a separable $\text{C}^*$-algebra whose primitive spectrum $X$ is compact Hausdorff and of finite dimension. Suppose that each primitive quotient $A(x)$ of $A$ is nuclear, purely infinite and stable. Then $A$ is isomorphic to $\text{C}(X) \otimes D$ for some $\text{KK}$-semiprojective stable Kirchberg algebra $D$ if and only if there is $\sigma \in \text{KK}(D, A)$ such that $\sigma_x \in \text{KK}(D, A(x))^{-1}$ for all $x \in X$. For any such $\sigma$ there is an isomorphism of $\text{C}(X)$-algebras $\Phi : \text{C}(X) \otimes D \to A$ such that $\text{KK}(\Phi|_D) = \sigma$.

We have an entirely similar result covering the unital case: Theorem 7.3. The required existence of $\sigma$ is a $\text{KK}$-theoretical analog of the classical condition of Fell that appears in the trivialization theorem of Dixmier and Douady [12] of continuous fields with fibers isomorphic to the compact operators. An important feature of our condition is that it is a priori much weaker than the condition that $A$ is $\text{KK}_{\text{C}(X)}$-equivalent to $\text{C}(X) \otimes D$. In particular, we do not need to worry at all about the potentially hard issue of constructing elements in $\text{KK}_{\text{C}(X)}(A, C(X) \otimes D)$. To illustrate this point, let us note that it is almost trivial to verify that the local existence of $\sigma$ is automatic.
for unital $C(X)$-algebras with fiber $O_n$ and hence to derive Theorem 1.1. A C*-algebra $D$ has the **automatic local triviality property** if any separable $C(X)$-algebra over a finite dimensional compact Hausdorff space $X$ all of whose fibers are isomorphic to $D$ is locally trivial. A unital C*-algebra $D$ has the **automatic local triviality property in the unital sense** if any separable unital $C(X)$-algebra over a finite dimensional compact Hausdorff space $X$ all of whose fibers are isomorphic to $D$ is locally trivial. The **automatic triviality property** is defined similarly.

**Theorem 1.3.** (Automatic triviality) A separable continuous $C(X)$-algebra over a finite dimensional compact Hausdorff space $X$ all of whose fibers are isomorphic to $O_2 \otimes K$ is isomorphic to $C(X) \otimes O_2 \otimes K$. The C*-algebra $O_2 \otimes K$ is the only Kirchberg algebra satisfying the automatic local triviality property and hence the automatic triviality property.

**Theorem 1.4.** (Automatic local triviality in the unital sense) A unital KK-semiprojective Kirchberg algebra $D$ has the automatic local triviality property in the unital sense if and only if all unital *-endomorphisms of $D$ are KK-equivalences. In that case, if $A$ is a separable unital $C(X)$-algebra over a finite dimensional compact Hausdorff space $X$ all of whose fibers are isomorphic to $D$, then $A \cong C(X) \otimes D$ if and only if there is $\sigma \in KK(D,A)$ such that the induced homomorphism $K_0(\sigma) : K_0(D) \to K_0(A)$ maps $[1_D]$ to $[1_A]$.

It is natural to ask if there are other unital Kirchberg algebras besides the Cuntz algebras which have the automatic local triviality property in the unital sense. Consider the following list $\mathcal{G}$ of pointed abelian groups:

(a) $\{\emptyset, 0\}$; (b) $(\mathbb{Z}, k)$ with $k > 0$;
(c) $(\mathbb{Z}/p^{s_1} \oplus \cdots \oplus \mathbb{Z}/p^{s_n}, p^{t_1} \oplus \cdots \oplus p^{t_n})$ where $p$ is a prime, $n \geq 1$, $0 \leq s_i < t_i$ for $1 \leq i \leq n$ and $0 < s_{i+1} - s_i < e_i + e_{i+1}$ for $1 \leq i < n$. If $n = 1$ the latter condition is vacuous. Note that if the integers $1 \leq e_1 \leq \cdots \leq e_n$ are given then there exists integers $s_1, \ldots, s_n$ satisfying the conditions above if and only if $e_{i+1} - e_i \geq 2$ for each $1 \leq i \leq n$. If that is the case one can choose $s_i = i - 1$ for $1 \leq i \leq n$.

(d) $(G(p_1) \oplus \cdots \oplus G(p_m), g_1 \oplus \cdots \oplus g_m)$ where $p_1, \ldots, p_m$ are distinct primes and each $(G(p_j), g_j)$ is a pointed group as in (c).

(e) $(\mathbb{Z} \oplus G(p_1) \oplus \cdots \oplus G(p_m), k \oplus g_1 \oplus \cdots \oplus g_m)$ where $(G(p_j), g_j)$ are as in (d). Moreover we require that $k > 0$ is divisible by $p_1^{s_{n(1)}+1} \cdots p_m^{s_{n(m)}+1}$ where $s_{n(j)}$ is defined as in (c) corresponding to the prime $p_j$.

**Theorem 1.5.** (Automatic local triviality in the unital sense – the UCT case) Let $D$ be a unital Kirchberg algebra which satisfies the UCT and has finitely generated $K$-theory groups. (i) $D$ has the automatic triviality property in the unital sense if and only if $D$ is isomorphic to either $O_2$ or $O_\infty$. (ii) $D$ has the automatic local triviality property in the unital sense if and only if $K_1(D) = 0$ and $(K_0(D), [1_D])$ is isomorphic to one of the pointed groups from the list $\mathcal{G}$. (iii) If $D$ is as in (ii), then a separable unital $C(X)$-algebra $A$ over a finite dimensional compact Hausdorff space $X$ all of whose fibers are isomorphic to $D$ is trivial if and only if there exists a homomorphism of groups $K_0(D) \to K_0(A)$ which maps $[1_D]$ to $[1_A]$.

We use semiprojectivity (in various flavors) to approximate and represent continuous $C(X)$-algebras as inductive limits of fibered products of $n$ locally trivial $C(X)$-subalgebras where $n \leq \dim(X) < \infty$. This clarifies the local structure of many $C(X)$-algebras (see Theorem 5.2) and
gives a new understanding of the K-theory of separable continuous $C(X)$-algebras with arbitrary nuclear fibers.

A remarkable isomorphism result for separable nuclear strongly purely infinite stable C*-algebras was announced (with an outline of the proof) by Kirchberg in [20]: two such C*-algebras $A$ and $B$ with the same primitive spectrum $X$ are isomorphic if and only if they are $KK_{C(X)}$-equivalent. This is always the case after tensoring with $O_2$. However the problem of recognizing when $A$ and $B$ are $KK_{C(X)}$-equivalent is open even for very simple spaces $X$ such as the unit interval or non-Hausdorff spaces with more than two points.

The proof of Theorem 4.6 (one of our main results) generalizes and refines a technique that was pioneered for fields over zero dimensional spaces in joint work with Pasnicu [11] and for fields over non-Hausdorff spaces with more than two points. The author is grateful to E. Blanchard, L. G. Brown and N. C. Phillips for useful discussions and comments.

2. $C(X)$-algebras

Let $X$ be a locally compact Hausdorff space. A $C(X)$-algebra is a C*-algebra $A$ endowed with a *-homomorphism $\theta$ from $C_0(X)$ to the center $ZM(A)$ of the multiplier algebra $M(A)$ of $A$ such that $C_0(X)A$ is dense in $A$; see [18], [3]. We write $fa$ rather than $\theta(f)a$ for $f \in C_0(X)$ and $a \in A$. If $Y \subseteq X$ is a closed set, we let $C_0(X,Y)$ denote the ideal of $C_0(X)$ consisting of functions vanishing on $Y$. Then $C_0(X,Y)A$ is a closed two-sided ideal of $A$ (by Cohen factorization). The quotient of $A$ by this ideal is a $C(X)$-algebra denoted by $A(Y)$ and is called the restriction of $A = A(X)$ to $Y$. The quotient map is denoted by $\pi_Y : A(X) \to A(Y)$. If $Z$ is a closed subset of $Y$ we have a natural restriction map $\pi_Y^Z : A(Y) \to A(Z)$ and $\pi_Z = \pi_Y^Z \circ \pi_Y$. If $Y$ reduces to a point $x$, we write $A(x)$ for $A(\{x\})$ and $\pi_x$ for $\pi_\{x\}$. The C*-algebra $A(x)$ is called the fiber of $A$ at $x$. The image $\pi_x(a) = A(x)$ of $a \in A$ is denoted by $a(x)$. A morphism of $C(X)$-algebras $\eta : A \to B$ induces a morphism $\eta_Y : A(Y) \to B(Y)$. If $A(x) \neq 0$ for $x$ in a dense subset of $X$, then $\theta$ is injective. If $X$ is compact, then $\theta(1) = 1_{M(A)}$. Let $A$ be a C*-algebra, $a \in A$ and $\mathcal{F}, \mathcal{G} \subseteq A$. Throughout the paper we will assume that $X$ is a compact Hausdorff space unless stated otherwise. If $\varepsilon > 0$, we write $a \in_{\varepsilon} \mathcal{F}$ if there is $b \in \mathcal{F}$ such that $\|a - b\| < \varepsilon$. Similarly, we write $\mathcal{F} \subseteq_{\varepsilon} \mathcal{G}$ if $a \in_{\varepsilon} \mathcal{G}$ for every $a \in \mathcal{F}$.

The following lemma collects some basic properties of $C(X)$-algebras.

**Lemma 2.1.** Let $A$ be a $C(X)$-algebra and let $B \subseteq A$ be a $C(X)$-subalgebra. Let $a \in A$ and let $Y$ be a closed subset of $X$.

(i) The map $x \mapsto \|a(x)\|$ is upper semi-continuous.

(ii) $\|\pi_Y(a)\| = \max\{\|\pi_x(a)\| : x \in Y\}$

(iii) If $a(x) \in \pi_x(B)$ for all $x \in X$, then $a \in B$.

(iv) If $\delta > 0$ and $a(x) \in_{\delta} \pi_x(B)$ for all $x \in X$, then $a \in_{\delta} B$.

(v) The restriction of $\pi_x : A \to A(x)$ to $B$ induces an isomorphism $B(x) \cong \pi_x(B)$ for all $x \in X$. 
Proof. (i), (ii) are proved in [3] and (iii) follows from (iv). (iv): By assumption, for each \( x \in X \), there is \( b_x \in B \) such that \( \| \pi_x(a - b_x) \| < \delta \). Using (i) and (ii), we find a closed neighborhood \( U_x \) of \( x \) such that \( \| \pi_U(x)(a - b_x) \| < \delta \). Since \( X \) is compact, there is a finite subcover \( (U_x) \). Let \( (\alpha_x) \) be a partition of unity subordinated to this cover. Setting \( b = \sum \alpha_x b_x \in B \), one checks immediately that \( \| \pi(x)(a - b) \| \leq \sum \alpha_x \| \pi_x(a - b_x) \| < \delta \), for all \( x \in X \). Thus \( \| a - b \| < \delta \) by (ii). (v): If \( \iota : B \hookrightarrow A \) is the inclusion map, then \( \pi_x(B) \) coincides with the image of \( \iota_x : B/C(X,x)B \rightarrow A/C(X,x)A \). Thus it suffices to check that \( \iota_x \) is injective. If \( \iota_x(b + C(X,x)B) = \pi_x(b) = 0 \) for some \( b \in B \), then \( b = fa \) for some \( f \in C(X,x) \) and some \( a \in A \). If \( (f_A) \) is an approximate unit of \( C(X,x) \), then \( b = \lim \lambda f_A a = \lim f_A b \) and hence \( b \in C(X,x)B \). 

A \( C(X) \)-algebra such that the map \( x \mapsto \|a(x)\| \) is continuous for all \( a \in A \) is called a continuous \( C(X) \)-algebra or a \( C^* \)-bundle [3], [23], [4]. A \( C^* \)-algebra \( A \) is a continuous \( C(X) \)-algebra if and only if \( A \) is the \( C^* \)-algebra of continuous sections of a continuous field of \( C^* \)-algebras over \( X \) in the sense of [12, Def. 10.3.1], (see [3], [4], [27]).

**Lemma 2.2.** Let \( A \) be a separable continuous \( C(X) \)-algebra over a locally compact Hausdorff space \( X \). If all the fibers of \( A \) are nonzero, then \( X \) has a countable basis of open sets. Thus the compact subspaces of \( X \) are metrizable.

**Proof.** Since \( A \) is separable, its primitive spectrum \( \text{Prim}(A) \) has a countable basis of open sets by [12, 3.3.4]. The continuous map \( \eta : \text{Prim}(A) \rightarrow X \) (induced by \( \theta : C_0(X) \rightarrow ZM(A) \cong C_0(\text{Prim}(A)) \)) is open since the \( C(X) \)-algebra \( A \) is continuous and surjective since \( A(x) \neq 0 \) for all \( x \in X \) (see [4, p. 388] and [27, Prop. 2.1, Thm. 2.3]).

**Lemma 2.3.** Let \( X \) be a compact metrizable space. A \( C(X) \)-algebra \( A \) all of whose fibers are nonzero and simple is continuous if and only if there is \( e \in A \) such that \( \|e(x)\| \geq 1 \) for all \( x \in X \).

**Proof.** By Lemma 2.1(i) it suffices to prove that \( \liminf_{n \to \infty} \|a(x_n)\| \geq \|a(x_0)\| \) for any \( a \in A \) and any sequence \( (x_n) \) converging to \( x_0 \) in \( X \). Set \( D = A(x_0) \) and let \( e \) be as in the statement. Let \( \psi : D \rightarrow A \) be a set-theoretical lifting of \( \text{id}_D \) such that \( \|\psi(d)\| = \|d\| \) for all \( d \in D \). Then \( \liminf_{n \to \infty} \|\pi_n,\psi(a(x_0)) - a(x_n)\| = 0 \) for all \( a \in A \), by Lemma 2.1(i). By applying this to \( e \), since \( \|e(x_n)\| \geq 1 \), we see that \( \liminf_{n \to \infty} \|\pi_n,\psi(e(x_0))\| \geq 1 \). Since \( D \) is a simple \( C^* \)-algebra, if \( \varphi_n : D \rightarrow B_n \) is a sequence of contractive maps such that \( \liminf_{n \to \infty} \|\varphi_n(\lambda d) - \lambda \varphi_n(d)\| = 0 \), \( \liminf_{n \to \infty} \|\varphi_n(c) - \varphi_n(c)^*\| = 0 \), for all \( c, d \in D \), \( \lambda \in \mathbb{C} \), and \( \liminf_{n \to \infty} \|\varphi_n(c)\| > 0 \) for some \( c \in D \), then \( \liminf_{n \to \infty} \|\varphi_n(c)\| = \|c\| \) for all \( c \in D \). In particular this observation applies to \( \varphi_n = \pi_{x_n} \psi \) by Lemma 2.1(i). Therefore

\[
\liminf_{n \to \infty} \|a(x_n)\| \geq \liminf_{n \to \infty} (\|\pi_n,\psi(a(x_0))\| - \|\pi_n,\psi(a(x_0)) - a(x_n)\|) = \|a(x_0)\|.
\]

Conversely, if \( A \) is continuous, take \( e \) to be a large multiple of some full element of \( A \). 

Let \( \eta : B \rightarrow A \) and \( \psi : E \rightarrow A \) be *-homomorphisms. The pullback of these maps is

\[
B \oplus_{\eta,\psi} E = \{(b, e) \in B \oplus E : \eta(b) = \psi(e)\}.
\]

We are going to use pullbacks in the context of \( C(X) \)-algebras. Let \( X \) be a compact space and let \( Y, Z \) be closed subsets of \( X \) such that \( X = Y \cup Z \). The following result is proved in [12, Prop. 10.1.13] for continuous \( C(X) \)-algebras.
Lemma 2.4. If \( A \) is a \( C(X) \)-algebra, then \( A \) is isomorphic to \( A(Y) \oplus_{\pi_Y} A(Z) \), the pullback of the restriction maps \( \pi_{Y \cap Z} : A(Y) \to A(Y \cap Z) \) and \( \pi_{Y \cap Z} : A(Z) \to A(Y \cap Z) \).

Proof. By the universal property of pullbacks, the maps \( \pi_Y \) and \( \pi_Z \) induce a map \( \eta : A \to A(Y) \oplus_{\pi_Y} A(Z) \), \( (\eta(a), \pi_Y(a), \pi_Z(a)) \), which is injective by Lemma 2.1(ii). Thus it suffices to show that the range of \( \eta \) is dense. Let \( b, c \in A \) be such that \( \pi_{Y \cap Z}(b - c) = 0 \) and let \( \varepsilon > 0 \). We shall find \( a \in A \) such that \( \|\eta(a) - (\pi_Y(b), \pi_Z(c))\| < \varepsilon \). By Lemma 2.1(i), there is an open neighborhood \( V \) of \( Y \cap Z \) such that \( \|\pi_x(b - c)\| < \varepsilon \) for all \( x \in V \). Let \( \{\lambda, \mu\} \) be a partition of unity on \( X \) subordinated to the open cover \( \{Y \cup V, Z \cup V\} \). Then \( a = \lambda b + \mu c \) is an element of \( A \) which has the desired property. \( \square \)

Let \( B \subset A(Y) \) and \( E \subset A(Z) \) be \( C(X) \)-subalgebras such that \( \pi_{Y \cap Z}^Y(E) \subseteq \pi_{Y \cap Z}^Y(B) \). As an immediate consequence of Lemma 2.4 we see that the pullback \( B \oplus_{\pi_{Y \cap Z}^Y} E \) is isomorphic to the \( C(X) \)-subalgebra \( B \oplus_{Y \cap Z} E \) of \( A \) defined as
\[
B \oplus_{Y \cap Z} E = \{a \in A : \pi_Y(a) \in B, \pi_Z(a) \in E\}.
\]

Lemma 2.5. The fibers of \( B \oplus_{Y \cap Z} E \) are given by
\[
\pi_x(B \oplus_{Y \cap Z} E) = \begin{cases} \pi_x(B), & \text{if } x \in X \setminus Z; \\ \pi_x(E), & \text{if } x \in Z, \end{cases}
\]
and there is an exact sequence of \( C^* \)-algebras
\[
0 \longrightarrow \{b \in B : \pi_{Y \cap Z}(b) = 0\} \longrightarrow B \oplus_{Y \cap Z} E \overset{\pi_Z}{\longrightarrow} E \longrightarrow 0
\]

Proof. Let \( x \in X \setminus Z \). The inclusion \( \pi_x(B \oplus_{Y \cap Z} E) \subseteq \pi_x(B) \) is obvious by definition. Given \( b \in B \), let us choose \( f \in C(X) \) vanishing on \( Z \) and such that \( f(x) = 1 \). Then \( a = (fb, 0) \) is an element of \( A \) by Lemma 2.4. Moreover \( a \in B \oplus_{Y \cap Z} E \) and \( \pi_x(a) = \pi_x(b) \). We have \( \pi_Z(B \oplus_{Y \cap Z} E) \subseteq E \), by definition. Conversely, given \( e \in E \), let us observe that \( \pi_{Y \cap Z}(a) = \pi_Y(B) \) (by assumption) and hence \( \pi_{Y \cap Z}(e) = \pi_{Y \cap Z}(b) \) for some \( b \in B \). Then \( a = (b, e) \) is an element of \( A \) by Lemma 2.4 and \( \pi(Z)(a, e) = e \). This completes the proof for the first part of the lemma and also it shows that the map \( \pi_Z \) from the sequence (1) is surjective. Its kernel is identified using Lemma 2.1(iii). \( \square \)

Let \( X, Y, Z \) and \( A \) be as above. Let \( \eta : B \to A(Y) \) be a \( C(Y) \)-linear *-monomorphism and let \( \psi : E \to A(Z) \) be a \( C(Z) \)-linear *-monomorphism. Assume that
\[
\pi_{Y \cap Z}(\psi(E)) \subseteq \pi_{Y \cap Z}(\eta(B)).
\]
This gives a map \( \gamma = \eta^{-1}_{Y \cap Z} \psi_{Y \cap Z} : E(Y \cap Z) \to B(Y \cap Z) \). To simplify notation we let \( \pi \) stand for both \( \pi_{Y \cap Z} \) and \( \pi_{Y \cap Z} \) in the following lemma.

Lemma 2.6. (a) There are isomorphisms of \( C(X) \)-algebras
\[
B \oplus_{\pi, \gamma} E \cong B \oplus_{\pi, \eta, \pi, \psi} E \cong \eta(B) \oplus_{Y \cap Z} \psi(E),
\]
where the second isomorphism is given by the map \( \chi : B \oplus_{\pi, \eta, \psi} E \to A \) induced by the pair \( \eta, \psi \). Its components \( \chi_x \) can be identified with \( \psi_x \) for \( x \in Z \) and with \( \eta_x \) for \( x \in X \setminus Z \).

(b) Condition (2) is equivalent to \( \psi(E) \subseteq \pi_Z(A \oplus_{Y \cap Z} \eta(B)) \).

(c) If \( F \) is a finite subset of \( A \) such that \( \pi_Y(F) \subset \varepsilon \eta(B) \) and \( \pi_Z(F) \subset \varepsilon \psi(E) \), then \( F \subset \varepsilon \eta(B) \oplus_{Y \cap Z} \psi(E) = \chi(B \oplus_{\pi, \eta, \psi} E) \).
Definition 2.7. Let $C$ be a class of C*-algebras. A $C(Z)$-algebra $E$ is called $C$-elementary if there is a finite partition of $Y$ into closed subsets $Z_1, \ldots, Z_r$ ($r \geq 1$) and there exist C*-algebras $D_1, \ldots, D_r$ in $C$ such that $E \cong \bigoplus_{i=1}^r C(Z_i) \otimes D_i$. The notion of category of a $C(X)$-algebra with respect to a class $C$ is defined inductively: if $A$ is $C$-elementary then $\text{cat}_C(A) = 0$; $\text{cat}_C(A) \leq n$ if there are closed subsets $Y$ and $Z$ of $X$ with $X = Y \cup Z$ and there exist a C(Y)-algebra $B$ such that $\text{cat}_C(B) \leq n - 1$, a $C$-elementary $C(Z)$-algebra $E$ and a *-monomorphism of $C(Y \cap Z)$-algebras $\gamma : E(Y \cap Z) \to B(Y \cap Z)$ such that $A$ is isomorphic to $B \oplus \pi_{\gamma \pi} E = \{(b, d) \in B \oplus E : \pi^Y_{\gamma \pi}(b) = \gamma \pi^Z_{\gamma \pi}(d)\}$.

By definition $\text{cat}_C(A) = n$ if $n$ is the smallest number with the property that $\text{cat}_C(A) \leq n$. If no such $n$ exists, then $\text{cat}_C(A) = \infty$.

Definition 2.8. Let $C$ be a class of C*-algebras and let $A$ be a $C(X)$-algebra. An $n$-fibered C-monomorphism $(\psi_0, \ldots, \psi_n)$ into $A$ consists of $(n+1)$ *-monomorphisms of $C(X)$-algebras $\psi_i : E_i \to A(Y_i)$, where $Y_0, \ldots, Y_n$ is a closed cover of $X$, each $E_i$ is a $C$-elementary $C(Y_i)$-algebra and

\[ \pi^Y_{Y_i \cap Y_j} \psi_i(E_i) \subseteq \pi^Y_{Y_i \cap Y_j} \psi_j(E_j), \quad \text{for all } i \leq j. \]

Given an $n$-fibered morphism into $A$ we have an associated continuous $C(X)$-algebra defined as the fibered product (or pullback) of the *-monomorphisms $\psi_i$:

\[ A(\psi_0, \ldots, \psi_n) = \{(d_0, \ldots, d_n) : d_i \in E_i, \pi^Y_{Y_i \cap Y_j} \psi_i(d_i) = \pi^Y_{Y_i \cap Y_j} \psi_j(d_j) \text{ for all } i, j\} \]

and an induced $C(X)$-monomorphism (defined by using Lemma 2.4)

\[ \eta = \eta(\psi_0, \ldots, \psi_n) : A(\psi_0, \ldots, \psi_n) \to A \subset \bigoplus_{i=0}^n A(Y_i), \]

\[ \eta(d_0, \ldots, d_n) = (\psi_0(d_0), \ldots, \psi_n(d_n)). \]

There are natural coordinate maps $p_i : A(\psi_0, \ldots, \psi_n) \to E_i$, $p_i(d_0, \ldots, d_n) = d_i$. Let us set $X_k = Y_k \cup \cdots \cup Y_n$. Then $(\psi_k, \ldots, \psi_n)$ is an $(n-k)$-fibered C-monomorphism into $A(X_k)$. Let $\eta_k : A(X_k)(\psi_k, \ldots, \psi_n) \to A(X_k)$ be the induced map and set $B_k = A(X_k)(\psi_k, \ldots, \psi_n)$. Let us note that $B_0 = A(\psi_0, \ldots, \psi_n)$ and that there are natural $C(X_{k-1})$-isomorphisms

\[ B_{k-1} \cong B_k \oplus \pi_{\psi_0, \ldots, \psi_{k-1}} \psi E_{k-1} \equiv B_k \oplus \pi_{\psi_0, \ldots, \psi_{k-1}} E_{k-1}. \]

where $\pi$ stands for $\pi_{X_{k-1} \cap Y_{k-1}}$ and $\gamma_k : E_{k-1}(X_k \cap Y_{k-1}) \to B_k(X_k \cap Y_{k-1})$ is defined by $(\gamma_k)_x = (\eta_k)_x^{-1}(\psi_{k-1})_x$, for all $x \in X_k \cap Y_{k-1}$. In particular, this decomposition shows that $\text{cat}_C(A(\psi_0, \ldots, \psi_n)) \leq n$.

Lemma 2.9. Suppose that the class $C$ from Definition 2.7 consists of stable Kirchberg algebras. If $A$ is a $C(X)$-algebra over a compact metrizable space $X$ such that $\text{cat}_C(A) < \infty$, then $A$ contains a full properly infinite projection and $A \cong A \otimes O_\infty \otimes K$. 

Proof. This is an immediate corollary of Lemmas 2.1, 2.4, 2.5. For illustration, let us verify (c). By assumption $\pi_x(\mathcal{F}) \subset \eta_x(B)$ for all $x \in X \setminus Z$ and $\pi_x(\mathcal{F}) \subset \psi_x(E)$ for all $x \in Z$. We deduce from Lemma 2.5 that $\pi_x(\mathcal{F}) \subset \pi_x(\eta(B) \oplus Y \cap Z \psi(E))$ for all $x \in X$. Therefore $\mathcal{F} \subset \eta(B) \oplus Y \cap Z \psi(E)$ by Lemma 2.1(iv).
Proof. We prove this by induction on \( n = \text{cat}_C(A) \). The case \( n = 0 \) is immediate since \( D \cong D \otimes \mathcal{O}_\infty \) for any Kirchberg algebra \( D \) [19], Let \( A = B \otimes_{\pi, \gamma} E \) where \( B, E \) and \( \gamma \) are as in Definition 2.7 with \( \text{cat}_C(B) = n - 1 \) and \( \text{cat}_C(E) = 0 \). Let us consider the exact sequence \( 0 \to J \to A \to E \to 0 \), where \( J = \{ b \in E : \pi_{Y \cap Z}(b) = 0 \} \). Since \( J \) is an ideal of \( B \cong B \otimes \mathcal{O}_\infty \otimes K \), \( J \) absorbs \( \mathcal{O}_\infty \otimes K \) by [22, Prop. 8.5]. Since both \( E \) and \( J \) are stable and purely infinite, it follows that \( A \) is stable by [30, Prop. 6.12] and purely infinite by [22, Prop. 3.5]. Since \( A \) has Hausdorff primitive spectrum, \( A \) is strongly purely infinite by [5, Thm. 5.8]. It follows that \( A \cong A \otimes \mathcal{O}_\infty \) by [22, Thm. 9.1]. Finally \( A \) contains a full properly infinite projection since there is a full embedding of \( \mathcal{O}_2 \) into \( A \) by [5, Prop. 5.6].

\[ \square \]

3. Semiprojectivity

In this section we study the notion of \( KK \)-semiprojectivity. The main result is Theorem 3.12. Let \( A \) and \( B \) be \( C^* \)-algebras. Two \(*\)-homomorphisms \( \varphi, \psi : A \to B \) are approximately unitarily equivalent, written \( \varphi \approx_u \psi \), if there is a sequence of unitaries \( (u_n) \) in the \( C^* \)-algebra \( B^+ = B + \mathbb{C}1 \) obtained by adjoining a unit to \( B \), such that \( \lim_{n \to \infty} \| u_n \varphi(a) u_n^* - \psi(a) \| = 0 \) for all \( a \in A \). We say that \( \varphi \) and \( \psi \) are asymptotically unitarily equivalent, written \( \varphi \approx_{ah} \psi \), if there is a norm continuous unitary valued map \( t \to u_t \in B^+ \), \( t \in [0,1] \), such that \( \lim_{t \to 0} \| u_t \psi(a) u_t^* - \psi(a) \| = 0 \) for all \( a \in A \). A \(*\)-homomorphism \( \varphi : D \to A \) is full if \( \varphi(d) \) is not contained in any proper two-sided closed ideal of \( A \) if \( d \in D \) is nonzero.

We shall use several times Kirchberg’s Theorem [29, Thm. 8.3.3] and the following theorem of Phillips [28].

**Theorem 3.1.** Let \( A \) and \( B \) be separable \( C^* \)-algebras such that \( A \) is simple and nuclear, \( B \cong B \otimes \mathcal{O}_\infty \), and there exist full projections \( p \in A \) and \( q \in B \). For any \( \sigma \in KK(A,B) \) there is a full \(*\)-homomorphism \( \varphi : A \to B \) such that \( KK(\varphi) = \sigma \). If \( K_0(\sigma)[p] = [q] \) then we may arrange that \( \varphi(p) = q \). If \( \psi : A \to B \) is another \(*\)-homomorphism such that \( KK(\psi) = KK(\varphi) \) and \( \psi(p) = q \), then \( \varphi \approx_{ah} \psi \) via a path of unitaries \( t \to u_t \in U(qBq) \).

Theorem 3.1 does not appear in this form in [28] but it is an immediate consequence of [28, Thm. 4.1.1]. Since \( pAp \otimes K \cong A \otimes K \) and \( qBq \otimes K \cong B \otimes K \) by [6], and \( qBq \otimes \mathcal{O}_\infty \cong qBq \) by [22, Prop. 8.5], it suffices to discuss the case when \( p \) and \( q \) are the units of \( A \) and \( B \). If \( \sigma \) is given, [28, Thm. 4.1.1] yields a full \(*\)-homomorphism \( \varphi : A \to B \otimes K \) such that \( KK(\varphi) = \sigma \). Let \( e \in K \) be a rank-one projection and suppose that \( [\varphi(1_A)] = [1_B \otimes e] \) in \( K_0(B) \). Since both \( \varphi(1_A) \) and \( 1_B \otimes e \) are full projections and \( B \cong B \otimes \mathcal{O}_\infty \), it follows by [28, Lemma 2.1.8] that \( u \varphi(1_A) u^* = 1_B \otimes e \) for some unitary in \( (B \otimes K)^+ \). Replacing \( \varphi \) by \( u \varphi u^* \) we can arrange that \( KK(\varphi) = \sigma \) and \( \varphi \) is unital. For the second part of the theorem let us note that any unital \(*\)-homomorphism \( \varphi : A \to B \) is full and if two unital \(*\)-homomorphisms \( \varphi, \psi : A \to B \) are asymptotically unitarily equivalent when regarded as maps into \( B \otimes K \), then \( \varphi \approx_{ah} \psi \) when regarded as maps into \( B \), by an argument from the proof of [28, Thm. 4.1.4].

A separable nonzero \( C^* \)-algebra \( D \) is semiprojective [1] if for any separable \( C^* \)-algebra \( A \) and any increasing sequence of two-sided closed ideals \( (J_n) \) of \( A \) with \( J = \bigcup_n J_n \), the natural map \( \lim \text{Hom}(D, A/J_n) \to \text{Hom}(D, A/J) \) (induced by \( \pi_n : A/J_n \to A/J \)) is surjective. If we weaken this condition and require only that the above map has dense range, where \( \text{Hom}(D, A/J) \) is given the point-norm topology, then \( D \) is called weakly semiprojective [14]. These definitions do not
change if we drop the separability of $A$. We shall use (weak) semiprojectivity in the following context. Let $A$ be a $C(X)$-algebra (with $X$ metrizable), let $x \in X$ and set $U_n = \{y \in X : d(y, x) \leq 1/n\}$. Then $J_n = C(X, U_n)A$ is an increasing sequence of ideals of $A$ such that $J = C(X, x)A$, $A/J_n \cong A(U_n)$ and $A/J \cong A(x)$.

Examples 3.2. (Weakly semiprojective C*-algebras) Any finite dimensional C*-algebra is semiprojective. A Kirchberg algebra $D$ satisfying the UCT and having finitely generated K-theory groups is weakly semiprojective by work of Neubüser [26], H. Lin [24] and Spielberg [32]. This also follows from Theorem 3.12 and Proposition 3.14 below. If in addition $K_1(D)$ is torsion free, then $D$ is semiprojective as proved by Spielberg [33] who extended the foundational work of Blackadar [1] and Szymanski [34].

The following generalizations of two results of Loring [25] are used in section 5; see [10].

Proposition 3.3. Let $D$ be a separable semiprojective C*-algebra. For any finite set $F \subseteq D$ and any $\varepsilon > 0$, there exist a finite set $G \subseteq D$ and $\delta > 0$ with the following property. Let $\pi : A \to B$ be a surjective $*$-homomorphism, and let $\varphi : D \to B$ and $\gamma : D \to A$ be $*$-homomorphisms such that $\|\pi\gamma(d) - \varphi(d)\| < \delta$ for all $d \in G$. Then there is a $*$-homomorphism $\psi : D \to A$ such that $\pi\psi = \varphi$ and $\|\gamma(c) - \psi(c)\| < \varepsilon$ for all $c \in F$.

Proposition 3.4. Let $D$ be a separable semiprojective C*-algebra. For any finite set $F \subseteq D$ and any $\varepsilon > 0$, there exist a finite set $G \subseteq D$ and $\delta > 0$ with the following property. For any two $*$-homomorphisms $\varphi, \psi : D \to B$ such that $\|\varphi(d) - \psi(d)\| < \delta$ for all $d \in G$, there is a homotopy $\Phi \in \text{Hom}(D, C[0, 1] \otimes B)$ such that $\Phi_0 = \varphi$ to $\Phi_1 = \psi$ and $\|\varphi(c) - \Phi_t(c)\| < \varepsilon$ for all $c \in F$ and $t \in [0, 1]$.

Definition 3.5. A separable C*-algebra $D$ is $KK$-stable if there is a finite set $G \subseteq D$ and there is $\delta > 0$ with the property that for any two $*$-homomorphisms $\varphi, \psi : D \to A$ such that $\|\varphi(a) - \psi(a)\| < \delta$ for all $a \in G$, one has $KK(\varphi) = KK(\psi)$.

Corollary 3.6. Any semiprojective C*-algebra is weakly semiprojective and $KK$-stable.

Proof. This follows from Proposition 3.4.

Proposition 3.7. Let $D$ be a separable weakly semiprojective C*-algebra. For any finite set $F \subseteq D$ and any $\varepsilon > 0$ there exist a finite set $G \subseteq D$ and $\delta > 0$ such that for any C*-algebras $B \subseteq A$ and any $*$-homomorphism $\varphi : D \to A$ with $\varphi(G) \subseteq B$, there is a $*$-homomorphism $\psi : D \to B$ such that $\|\varphi(c) - \psi(c)\| < \varepsilon$ for all $c \in F$. If in addition $D$ is $KK$-stable, then we can choose $G$ and $\delta$ such that we also have $KK(\psi) = KK(\varphi)$.

Proof. This follows from [14, Thms. 3.1, 4.6]. Since the result is essential to us we include a short proof. Fix $F$ and $\varepsilon$. Let $(G_n)$ be an increasing sequence of finite subsets of $D$ whose union is dense in $D$. If the statement is not true, then there are sequences of C*-algebras $C_n \subseteq A_n$ and $*$-homomorphisms $\varphi_n : D \to A_n$ satisfying $\varphi_n(G_n) \subseteq 1/n C_n$ and with the property that for any $n \geq 1$ there is no $*$-homomorphism $\psi_n : D \to C_n$ such that $\|\varphi_n(c) - \psi_n(c)\| < \varepsilon$ for all $c \in F$. Set $B_i = \prod_{n \geq i} A_n$ and $E_i = \prod_{n \geq i} C_n \subseteq B_i$. If $\nu_i : B_i \to B_{i+1}$ is the natural projection, then $\nu_i(E_i) = E_{i+1}$. Let us observe that if we define $\Phi_i : D \to B_i$ by $\Phi_i(d) = (\varphi_i(d), \varphi_{i+1}(d), \ldots)$, then the image of $\Phi = \lim_i \Phi_i : D \to \lim_i (B_i, \nu_i)$ is contained in $\lim_i (E_i, \nu_i)$. Since $D$ is weakly semiprojective,
Proposition 3.8. Let $D$ be a separable semiprojective C*-algebra. For any finite set $F \subset D$ and any $\varepsilon > 0$, there exist a finite set $G \subset D$ and $\delta > 0$ with the following property. Let $\pi : A \to B$ be a surjective *-homomorphism which maps a C*-subalgebra $A'$ of $A$ onto a C*-subalgebra $B'$ of $B$. Let $\varphi : D \to B'$ and $\gamma : D \to A$ be *-homomorphisms such that $\gamma(G) \subset A'$ and $\|\pi\gamma(d) - \varphi(d)\| < \delta$ for all $d \in G$. Then there is a *-homomorphism $\psi : D \to A'$ such that $\pi \psi = \varphi$ and $\|\psi(c) - \psi(c)\| < \varepsilon$ for all $c \in F$.

Proof. Let $G_L$ and $\delta_L$ be given by Proposition 3.3 applied to the input data $F$ and $\varepsilon/2$. We may assume that $F \subset G_L$ and $\varepsilon > \delta_L$. Next, let $G_P$ and $\delta_P$ be given by Proposition 3.7 applied to the input data $G_L$ and $\delta_L/2$. We show now that $G := G_L \cup G_P$ and $\delta := \min\{\delta_P, \delta_L/2\}$ have the desired properties. We have $\gamma(G_P) \subset \delta P \subset G$ and $\delta \leq \delta_L$. By Proposition 3.7 there is a *-homomorphism $\gamma' : D \to A'$ such that $\|\gamma'(d) - \gamma(d)\| < \delta_L/2$ for all $d \in G_L$. Then

$$\|\pi\gamma'(d) - \varphi(d)\| \leq \|\pi\gamma'(d) - \gamma(d)\| + \|\gamma(d) - \varphi(d)\| < \delta_L/2 + \delta \leq \delta_L$$

for all $d \in G_L$ since $G_L \subset G$ and $\delta \leq \delta_L/2$. Therefore we can invoke Proposition 3.3 to perturb $\gamma'$ to a *-homomorphism $\psi : D \to A'$ such that $\pi \psi = \varphi$ and $\|\gamma'(d) - \psi(d)\| < \varepsilon/2$ for all $d \in F$. Finally we observe that for $d \in F \subset G_L$

$$\|\gamma(d) - \psi(d)\| \leq \|\gamma(d) - \gamma'(d)\| + \|\gamma'(d) - \psi(d)\| < \delta_L/2 + \varepsilon/2 < \varepsilon.$$

Definition 3.9. (a) A separable C*-algebra $D$ is KK-semiprojective if for any separable C*-algebra $A$ and any increasing sequence of two-sided closed ideals $(J_n)$ of $A$ with $J = \overline{\bigcup_n J_n}$, the natural map $\lim_n KK(D, A/J_n) \to KK(D, A/J)$ is surjective.

(b) We say that the functor $KK(D, -)$ is continuous if for any inductive system $B_1 \to B_2 \to \ldots$ of separable C*-algebras, the induced map $\lim_n KK(D, B_n) \to KK(D, \lim_n B_n)$ is bijective.

Proposition 3.10. Any separable KK-semiprojective C*-algebra is KK-stable.

Proof. We shall prove the statement by contradiction. Let $D$ be separable KK-semiprojective C*-algebra. Let $(G_n)$ be an increasing sequence of finite subsets of $D$ whose union is dense in $D$. If the statement is not true, then there are sequences of *-homomorphisms $\varphi_n, \psi_n : D \to A_n$ such that $\|\varphi_n(d) - \psi_n(d)\| < 1/n$ for all $d \in G_n$ and yet $KK(\varphi_n) \neq KK(\psi_n)$ for all $n \geq 1$. Set $B_i = \prod_{n \geq i} A_n$ and let $\nu_i : B_i \to B_{i+1}$ be the natural projection. Let us define $\Phi_i, \Psi_i : D \to B_i$ by $\Phi_i(d) = (\varphi_1(d), \varphi_{i+1}(d), \ldots)$ and $\Psi_i(d) = (\psi_1(d), \psi_{i+1}(d), \ldots)$, for all $d \in D$. Let $B'_i$ be the separable C*-subalgebra of $B_i$ generated by the images of $\Phi_i$ and $\Psi_i$. Then $\nu_i(B'_i) = B'_{i+1}$ and one verifies immediately that $\lim \Phi_i = \lim \Psi_i : D \to \lim (B'_i, \nu_i)$. Since $D$ is KK-semiprojective, we must have $KK(\Phi_i) = KK(\Psi_i)$ for some $i$ and hence $KK(\varphi_n) = KK(\psi_n)$ for all $n \geq i$. This gives a contradiction.

Proposition 3.11. A unital Kirchberg algebra $D$ is KK-stable if and only if $D \otimes K$ is KK-stable. $D$ is weakly semiprojective if and only if $D \otimes K$ is weakly semiprojective.
Proof. Since \( KK(D, A) \cong KK(D, A \otimes K) \cong KK(D \otimes K, A \otimes K) \) the first part of the proposition is immediate. Suppose now that \( D \otimes K \) is weakly semiprojective. Then \( D \) is weakly semiprojective as shown in the proof of [32, Thm. 2.2]. Conversely, assume that \( D \) is weakly semiprojective. It suffices to find \( \alpha \in \text{Hom}(D \otimes K, D) \) and a sequence \( (\beta_n) \) in \( \text{Hom}(D, D \otimes K) \) such that \( \beta_n \alpha \) converges to \( \text{id}_{D \otimes K} \).

Let \( s_i \) be the canonical generators of \( O_\infty \). If \( (e_{ij}) \) is a system of matrix units for \( K \), then \( \lambda(e_{ij}) = s_i^* s_j \) defines a \( * \)-homomorphism \( K \to O_\infty \) such that \( KK(\lambda) \in KK(K, O_\infty)^{-1} \). Therefore, by composing \( \text{id}_D \otimes \lambda \) with some isomorphism \( D \otimes O_\infty \cong D \) (given by [29, Thm. 7.6.6]) we obtain a \( * \)-monomorphism \( \alpha : D \otimes K \to D \) which induces a \( KK \)-equivalence. Let \( \beta : D \to D \otimes K \) be defined by \( \beta(d) = d \otimes e_{11} \). Then \( \beta \alpha \in \text{End}(D \otimes K) \) induces a \( KK \)-equivalence and hence after replacing \( \beta \) by \( \theta \beta \) for some automorphism \( \theta \) of \( D \otimes K \), we may arrange that \( KK(\beta \alpha) = KK(\text{id}_D) \).

By Theorem 3.1, \( \beta \alpha \approx \text{id}_{D \otimes K} \), so that there is a sequence of unitaries \( u_n \in (D \otimes K)^+ \) such that \( u_n \beta \alpha(-)u_n^* \) converges to \( \text{id}_{D \otimes K} \). \( \square \)

Theorem 3.12. For a separable \( C^* \)-algebra \( D \) consider the following properties:

(i) \( D \) is \( KK \)-semiprojective.

(ii) The functor \( KK(D, -) \) is continuous.

(iii) \( D \) is weakly semiprojective and \( KK \)-stable.

Then (i) \( \Leftrightarrow \) (ii). Moreover, (iii) \( \Rightarrow \) (i) if \( D \) is nuclear and (i) \( \Rightarrow \) (iii) if \( D \) is a Kirchberg algebra. Thus (i) \( \Leftrightarrow \) (ii) \( \Leftrightarrow \) (iii) for any Kirchberg algebra \( D \).

Proof. The implication (ii) \( \Rightarrow \) (i) is obvious. (i) \( \Rightarrow \) (ii): Let \( (B_n, \gamma_{n,m}) \) be an inductive system with inductive limit \( B \) and let \( \gamma_n : B_n \to B \) be the canonical maps. We have an induced map \( \beta : \varinjlim KK(D, B_n) \to KK(D, B) \). First we show that \( \beta \) is surjective. The mapping telescope construction of L. G. Brown (as described in the proof of [1, Thm. 3.1]) produces an inductive system of \( C^* \)-algebras \( (T_n, \eta_{n,m}) \) with inductive limit \( B \) such that each \( \eta_{n,n+1} \) is surjective, and each canonical map \( \eta_n : T_n \to B \) is homotopic to \( \gamma_n \alpha_n \) for some \( * \)-homomorphism \( \alpha_n : T_n \to B_n \). In particular \( KK(\eta_n) = KK(\gamma_n)KK(\alpha_n) \). Let \( x \in KK(D, B) \). By (i) there are \( n \) and \( y \in KK(D, T_n) \) such that \( KK(\eta_n)y = x \) and hence \( KK(\gamma_n)KK(\alpha_n)y = x \). Thus \( z = KK(\alpha_n)y \in KK(D, B_n) \) is a lifting of \( x \). Let us show now that the map \( \beta \) is injective. Let \( x \) be an element in the kernel of the map \( KK(D, B_n) \to KK(D, B) \). Consider the commutative diagram whose exact rows are portions of the Puppe sequence in \( KK \)-theory [2, Thm. 19.4.3] and with vertical maps induced by \( \gamma_m : B_m \to B \), \( m \geq n \).

\[
\begin{array}{ccc}
KK(D, C_{\gamma_n}) & \longrightarrow & KK(D, B_n) \longrightarrow KK(D, B) \\
\downarrow & & \downarrow \\
KK(D, C_{\gamma_n}) & \longrightarrow & KK(D, B_n) \longrightarrow KK(D, B_m)
\end{array}
\]

By exactness, \( x \) is the image of some element \( y \in KK(D, C_{\gamma_n}) \). Since \( C_{\gamma_n} = \varinjlim C_{\gamma_{n,m}} \), the map \( \varinjlim KK(D, C_{\gamma_n}) \to KK(D, C_{\gamma_n}) \) is surjective by the first part of the proof. Therefore there is \( m \geq n \) such that \( y \) lifts to some \( z \in KK(D, C_{\gamma_{n,m}}) \). The image of \( z \) in \( KK(D, B_m) \) equals \( KK(\gamma_{n,m})x \) and vanishes by exactness of the bottom row.

(iii) \( \Rightarrow \) (i): Let \( A, (J_n) \) and \( J \) be as in Definition 3.9. Using the five-lemma and the split exact sequence \( 0 \to KK(D, A) \to KK(D, A^+) \to KK(D, C) \to 0 \), we reduce the proof to the case when \( A \) is unital. Let \( x \in KK(D, A/J) \). Since the map \( KK(D^+, A/J) \to KK(D, A/J) \) is
such that

Proposition 3.14. generated $K$-theory groups $[29, Prop. 8.4.15]$. A similar argument gives the following:

□

This is very similar to the proof of the implication (iii) from $D$ such that

If $\psi : D \to A/J_n \otimes \mathcal{O}_\infty \otimes K$ such that $\|\pi_n \psi(d) - \varphi(d)\| < \delta$ for all $d \in \mathcal{G}$, where $\mathcal{G}$ and $\delta$ are as in the definition of $KK$-stability. Therefore $KK(\pi_n \psi) = KK(\varphi)$ and hence $KK(\psi)$ is a lifting of $x$ to $KK(D, A/J_n)$.

(i) $\Rightarrow$ (iii): $D$ is $KK$-stable by Proposition 3.10. It remains to show that $D$ is weakly semiprojective. Since any nonunital Kirchberg algebra is isomorphic to the stabilization of a unital one (see [29, Prop. 4.1.3]) and since by Proposition 3.11 $D$ is $KK$-semiprojective if and only if $D \otimes K$ is $KK$-semiprojective, we may assume that $D$ is unital. Let $A_n, (J_n), \pi_m,n : A/J_m \to A/J_n (m \leq n)$ and $\pi_n : A/J_n \to A/J$ be as in the definition of weak semiprojectivity. By [1, Cor. 2.15], we may assume that $A$ and the $*$-homomorphism $\varphi : D \to A$ (for which we want to construct an approximating lifting) are unital. In particular $\varphi$ is injective since $D$ is simple. Set $B = \varphi(D) \subset A/J$ and $B_n = \pi_n^{-1}(B) \subset A/J_n$. The corresponding maps $\pi_m,n : B_m \to B_n (m \leq n)$ and $\pi_n : B_n \to B$ are surjective and hence induce an isomorphism $\lim_{n \to \infty} (B_n, \pi_{n,n+1}) \cong B$.

Given $\varepsilon > 0$ and $F \subset D$ (a finite set) we are going to produce an approximate lifting $\varphi_n : D \to B_n$ for $\varphi$. Since $1_B$ is a properly infinite projection, it follows by [1, Props. 2.18 and 2.23] that the unit $1_n$ of $B_n$ is a properly infinite projection, for all sufficiently large $n$. Since $D$ is $KK$-semiprojective, there exist $m$ and an element $h \in KK(D, B_m)$ which lifts $KK(\varphi)$ such that $K_0(h)[1_J] = [1_m]$. By [29, Thm. 8.3.3], there is a full $*$-homomorphism $\eta : D \to B_m \otimes K$ such that $KK(\eta) = h$. By [29, Prop. 4.1.4], since both $\eta(1_D)$ and $1_m$ are full and properly infinite projections in $B_m \otimes K$, there is a partial isometry $w \in B_m \otimes K$ such that $w^*w = \eta(1_D)$ and $ww^* = 1_m$. Replacing $\eta$ by $w \eta(-) w^*$, we may assume that $\eta : D \to B_m$ is unital. Then $KK(\pi_m \eta) = KK(\pi_m)h = KK(\varphi)$. By Theorem 3.1, $\pi_m \eta \approx_{nh} \varphi$. Thus there is a unitary $u \in B$ such that $\|u \pi_m \eta(d)u^* - \varphi(d)\| < \varepsilon$ for all $d \in F$. Since $C(T)$ is semiprojective, there is $n \geq m$ such that $u$ lifts to a unitary $u_n \in B_n$. Then $\varphi_n = u_n \pi_m \eta(-) u_n^*$ is a $*$-homomorphism from $D$ to $B_n$ such that $\|\pi_n \varphi_n(d) - \varphi(d)\| < \varepsilon$ for all $d \in F$.

□

Corollary 3.13. Any separable nuclear semiprojective $C^*$-algebra is $KK$-semiprojective.

Proof. This is very similar to the proof of the implication (iii) $\Rightarrow$ (i) of Theorem 3.12. Alternatively, the statement follows from Corollary 3.6 and Theorem 3.12.

□

Blackadar has shown that a semiprojective Kirchberg algebra satisfying the UCT has finitely generated $K$-theory groups [29, Prop. 8.4.15]. A similar argument gives the following:

Proposition 3.14. Let $D$ be a separable $C^*$-algebra satisfying the UCT. Then $D$ is $KK$-semiprojective if and only $K_*(D)$ is finitely generated.

Proof. If $K_*(D)$ is finitely generated, then $D$ is $KK$-semiprojective by [31]. Conversely, assume that $D$ is $KK$-semiprojective. Since $D$ satisfies the UCT, we infer that if $G = K_i(D)$ $(i = 0, 1)$, then $G$ is semiprojective in the category of countable abelian groups, in the sense that if $H_1 \to H_2 \to \cdots$ is an inductive system of countable abelian groups with inductive limit $H$, then the natural map $\lim_{\to 0} \Hom(G, H_0) \to \Hom(G, H)$ is surjective. This implies that $G$ is finitely generated. Indeed, taking $H = G$, we see that $id_G$ lifts to $\Hom(G, H_n)$ for some finitely generated subgroup $H_n$ of $G$ and hence $G$ is a quotient of $H_n$.

□
4. Approximation of \((C(X))\)-algebras

In this section we use weak semiprojectivity to approximate a \((C(X))\)-algebra \(A\) by \((C(X))\)-subalgebras given by pullbacks of \(n\)-fibered monomorphisms into \(A\).

**Lemma 4.1.** Let \(D\) be a finite direct sum of simple \(C^\ast\)-algebras and let \(\varphi, \psi : D \to A\) be \(*\)-homomorphisms. Suppose that \(\mathcal{H} \subset D\) contains a nonzero element from each simple direct summand of \(D\). If \(\|\psi(d) - \varphi(d)\| \leq \|d\|/2\) for all \(d \in \mathcal{H}\), then \(\varphi\) is injective if and only if \(\psi\) is injective.

**Proof.** Let us note that \(\varphi\) is injective if and only if \(\|\varphi(d)\| = \|d\|\) for all \(d \in \mathcal{H}\). Therefore if \(\varphi\) is injective, then \(\|\psi(d)\| \geq \|\psi(d) - \varphi(d)\| \geq \|d\|/2\) for all \(d \in \mathcal{H}\) and hence \(\psi\) is injective. \(\Box\)

A sequence \((A_n)\) of subalgebras of \(A\) is called exhaustive if for any finite subset \(\mathcal{F}\) of \(A\) and any \(\varepsilon > 0\) there is \(n\) such that \(\mathcal{F} \subset \varepsilon A_n\).

**Lemma 4.2.** Let \(\mathcal{C}\) be a class consisting of finite direct sums of separable simple weakly semiprojective \(C^\ast\)-algebras. Let \(X\) be a compact metrizable space and let \(A\) be a \((C(X))\)-algebra. Let \(\mathcal{F} \subset A\) be a finite set, let \(\varepsilon > 0\) and suppose that \(A(x)\) admits an exhaustive sequence of \(C^\ast\)-algebras isomorphic to \(C^\ast\)-algebras in \(\mathcal{C}\) for some \(x \in X\). Then there exist a compact neighborhood \(U\) of \(x\) and a \(*\)-homomorphism \(\varphi : D \to A(U)\) for some \(D \in \mathcal{C}\) such that \(\pi_U(\mathcal{F}) \subset \varepsilon \varphi(D)\). If \(A\) is a continuous \((C(X))\)-algebra, then we may arrange that \(\varphi_z\) is injective for all \(z \in U\).

**Proof.** Let \(\mathcal{F} = \{a_1, \ldots, a_r\}\) and \(\varepsilon\) be given. By hypothesis there exist \(D \in \mathcal{C}\), \(\{c_1, \ldots, c_r\} \subset D\) and a \(*\)-monomorphism \(\iota : D \to A(x)\) such that \(\|\pi_x(a_i) - \iota(c_i)\| < \varepsilon/2\), for all \(i = 1, \ldots, r\). Set \(U_n = \{y \in X : d(x, y) \leq 1/n\}\). Choose a full element \(d_j\) in each direct summand of \(D\). Since \(D\) is weakly semiprojective, there is a \(*\)-homomorphism \(\varphi : D \to A(U_n)\) (for some \(n\)) such that \(\|\pi_x(\varphi(c_i)) - \iota(c_i)\| < \varepsilon/2\) for all \(i = 1, \ldots, r\), and \(\|\pi_x(\varphi(d_j)) - \iota(d_j)\| \leq \|d_j\|/2\) for all \(d_j\). Therefore

\[\|\pi_x(\varphi(c_i)) - \pi_x(a_i)\| = \|\pi_x(\varphi(c_i) - a_i)\| < \varepsilon/2 + \varepsilon/2 = \varepsilon\]

and \(\varphi_z\) is injective by Lemma 4.1. By Lemma 2.1(i), after increasing \(n\) and setting \(U = U_n\) and \(\varphi = \pi_U \varphi\), we have

\[\|\varphi(c_i) - \pi_U(a_i)\| = \|\pi_U(\varphi(c_i) - a_i)\| < \varepsilon,\]

for all \(i = 1, \ldots, r\). This shows that \(\pi_U(\mathcal{F}) \subset \varepsilon \varphi(D)\). If \(A\) is continuous, then after shrinking \(U\) we may arrange that \(\|\varphi_z(d_j)\| \geq \|\varphi_z(d_j)\|/2 = \|d_j\|/2\) for all \(d_j\) and all \(z \in U\). This implies that \(\varphi_z\) in injective for all \(z \in U\). \(\Box\)

**Lemma 4.3.** Let \(X\) be a compact metrizable space and let \(A\) be a separable continuous \((C(X))\)-algebra the fibers of which are stable Kirchberg algebras. Let \(\mathcal{F} \subset A\) be a finite set and let \(\varepsilon > 0\).

Suppose that there exist a KK-semiprojective stable Kirchberg algebra \(D\) and \(\sigma \in KK(D, A)\) such that \(\sigma_x \in KK(D, A(x))^{-1}\) for some \(x \in X\). Then there exist a closed neighborhood \(U\) of \(x\) and a full \(*\)-homomorphism \(\psi : D \to A(U)\) such that \(KK(\psi) = \sigma_U\) and \(\pi_U(\mathcal{F}) \subset \varepsilon \psi(D)\).

**Proof.** By [29, Thm. 8.4.1] there is an isomorphism \(\psi_0 : D \to A(x)\) such that \(KK(\psi_0) = \sigma_x\). Let \(\mathcal{H} \subset D\) be such that \(\psi_0(\mathcal{H}) = \pi_x(\mathcal{F})\). Set \(U_n = \{y \in X : d(x, y) \leq 1/n\}\). By Theorem 3.12 \(D\) is KK-stable and weakly semiprojective. By Proposition 3.7 there exists a \(*\)-homomorphism \(\psi_n : D \to A(U_n)\) (for some \(n\)) such that \(\|\pi_x(\psi_n(d) - \psi_0(d))\| < \varepsilon\) for all \(d \in \mathcal{H}\) and \(KK(\pi_x \psi_n) = \sigma_x\) and \(\psi_n(\mathcal{H}) = \pi_x(\mathcal{F})\). Therefore there is an isomorphism \(\psi_n \psi^{-1} : A(U_n) \to A(U)\) such that \(KK(\psi_n \psi^{-1}) = KK(\psi)\). Then there exist a compact neighborhood \(U\) of \(x\) and a \(*\)-homomorphism \(\psi : D \to A(U)\) such that \(\psi_n \psi^{-1} \pi_U(\mathcal{F}) \subset \varepsilon \psi(D)\). Therefore \(\psi\) is a \(*\)-homomorphism \(\psi : D \to A(U)\) such that \(\pi_U(\mathcal{F}) \subset \varepsilon \psi(D)\). That is, \(KK(\psi) = \sigma_U\) and \(\pi_U(\mathcal{F}) \subset \varepsilon \psi(D)\). \(\Box\)
\( \text{KK}(\psi_0) = \sigma_x \). Since \( \lim m \to \infty \text{KK}(D, A(U_m)) = \text{KK}(D, A(x)) \), we deduce that there is \( m \geq n \) such that \( \text{KK}(\pi_{U_m} \psi_n) = \sigma_{U_m} \). By increasing \( m \) we may arrange that \( \pi_{U_m}(F) \subset \varepsilon \pi_{U_m}(\psi_n(D)) \) since we have seen that \( \pi_x(F) = \psi_0(H) \subset \varepsilon \pi_x(\psi_n(D)) \). We can arrange that \( \psi_z \) is injective for all \( z \in U \) by reasoning as in the proof of Lemma 4.2. We conclude by setting \( U = U_m \) and \( \psi = \pi_{U_m} \psi_n \).

The following lemma is useful for constructing fibered morphisms.

**Lemma 4.4.** Let \( \{D_j\}_{j \in J} \) be a finite family consisting of finite direct sums of weakly semiprojective simple \( \text{C}^* \)-algebras. Let \( \varepsilon > 0 \) and for each \( j \in J \) let \( \mathcal{H}_j \subset D_j \) be a finite set such that for each direct summand of \( D_j \) there is an element of \( \mathcal{H}_j \) of norm \( \geq \varepsilon \) which is contained and is full in that summand. Let \( \mathcal{G}_j \subset D_j \) and \( \delta_j > 0 \) be given by Proposition 3.7 applied to \( D_j, \mathcal{H}_j \) and \( \varepsilon/2 \). Let \( X \) be a compact metrizable space, let \( \{Z_j\}_{j \in J} \) be disjoint nonempty closed subsets of \( X \) and let \( Y = \bigcup \{Z_j\}_{j \in J} \). Let \( A \) be a continuous \( C(X) \)-algebra and let \( \mathcal{F} \) be a finite subset of \( A \). Let \( \cdot : B(Y) \to A(Y) \) be a \( \ast \)-monomorphism of \( C(Y) \)-algebras and let \( \varphi_j : D_j \to A(Z_j) \) be \( \ast \)-homomorphisms such that \( (\varphi_j)_x \) is injective for all \( x \in Z_j \) and \( j \in J \), and which satisfy the following conditions:

(i) \( \pi_{Z_j}(\mathcal{F}) \subset_{\varepsilon/2} \varphi_j(\mathcal{H}_j) \), for all \( j \in J \),

(ii) \( \pi_Y(\mathcal{F}) \subset_{\varepsilon} \eta(B) \),

(iii) \( \pi_{Y \cap Z_j}(\mathcal{G}_j) \subset_{\varepsilon} \pi_Y(\mathcal{H}_j) \), for all \( j \in J \).

Then, there are \( C(Z_j) \)-linear \( \ast \)-monomorphisms \( \psi_j : C(Z_j) \otimes D_j \to A(Z_j) \), satisfying

\[
(6) \quad \|\varphi_j(c) - \psi_j(c)\| < \varepsilon/2, \quad \text{for all } c \in \mathcal{H}_j, \text{ and } j \in J,
\]

and such that if we set \( E = \bigoplus_j C(Z_j) \otimes D_j, \) \( Z = \bigcup_j Z_j, \) and \( \psi : E \to A(Z) = \bigoplus_j A(Z_j), \)

\[
\psi = \oplus_j \psi_j, \text{ then } \pi_{Y \cap Z}(\psi(E)) \subset \pi_Y(\eta(B)), \pi_Z(\mathcal{F}) \subset_{\varepsilon} \psi(E) \text{ and hence }
\]

\[
\mathcal{F} \subset_{\varepsilon} \eta(B) \oplus_{Y \cap Z} \psi(E) = \chi(B \oplus_{Y \cap Z} \eta(E)),
\]

where \( \chi \) is the isomorphism induced by the pair \((\eta, \psi)\). If we assume that each \( D_j \) is KK-stable, then we also have \( \text{KK}(\varphi_j) = \text{KK}(\psi_j(D_j)) \) for all \( j \in J \).

**Proof.** Let \( \mathcal{F} = \{a_1, \ldots, a_r\} \subset A \) be as in the statement. By (i), for each \( j \in J \) we find \( \{c_{1i}^{(j)}, \ldots, c_{ri}^{(j)}\} \subset \mathcal{H}_j \) such that \( \|\varphi_j(c_{1i}^{(j)}) - \pi_{Z_j}(a_i)\| < \varepsilon/2 \) for all \( i \). Consider the \( C(X) \)-algebra \( A \oplus_{Y \cap Z} \eta(B) \subset A \). From (iii), Lemma 2.1(iv) and Lemma 2.5 we obtain

\[
\varphi_j(\mathcal{G}_j) \subset_{\varepsilon} \pi_{Z_j}(A \oplus_{Y \cap Z} \eta(B)).
\]

Applying Proposition 3.7 we perturb \( \varphi_j \) to a \( \ast \)-homomorphism \( \psi_j : D_j \to \pi_{Z_j}(A \oplus_{Y \cap Z} \eta(B)) \) satisfying (6), and hence such that \( \|\varphi_j(c_i^{(j)}) - \psi_j(c_i^{(j)})\| < \varepsilon/2 \), for all \( i, j \). Therefore

\[
\|\psi_j(c_i^{(j)}) - \pi_{Z_j}(a_i)\| \leq \|\varphi_j(c_i^{(j)}) - \psi_j(c_i^{(j)})\| + \|\psi_j(c_i^{(j)}) - \pi_{Z_j}(a_i)\| < \varepsilon.
\]

This shows that \( \pi_{Z_j}(\mathcal{F}) \subset_{\varepsilon} \psi_j(D_j) \). From (6) and Lemma 4.1 we obtain that each \( (\psi_j)_x \) is injective. We extend \( \psi_j \) to a \( C(Z_j) \)-linear \( \ast \)-monomorphism \( \psi_j : C(Z_j) \otimes D_j \to \pi_{Z_j}(A \oplus_{Y \cap Z} \eta(B)) \) and then we define \( E, \psi \) and \( Z \) as in the statement. In this way we obtain that \( \psi : E \to (A \oplus_{Y \cap Z} \eta(B))(Z) \subset A(Z) \) satisfies

\[
(7) \quad \pi_Z(\mathcal{F}) \subset_{\varepsilon} \psi(E).
\]

The property \( \psi(E) \subset (A \oplus_{Y \cap Z} \eta(B))(Z) \) is equivalent to \( \pi_{Y \cap Z}^Z(\psi(E)) \subset \pi_{Y \cap Z}(\eta(B)) \) by Lemma 2.6(b). Finally, from (ii), (7) and Lemma 2.6(c) we get \( \mathcal{F} \subset_{\varepsilon} \eta(B) \oplus_{Y \cap Z} \psi(E) \).
Let $\mathcal{C}$ be as in Lemma 4.2. Let $A$ be a $C(X)$-algebra, let $\mathcal{F} \subset A$ be a finite set and let $\varepsilon > 0$. An $(\mathcal{F}, \varepsilon, \mathcal{C})$-approximation of $A$

\begin{equation}
\alpha = \{\mathcal{F}, \varepsilon, \{U_i, \varphi_i : D_i \to A(U_i), \mathcal{H}_i, \mathcal{G}_i, \delta_i\}_{i \in I}\},
\end{equation}

is a collection with the following properties: $(U_i)_{i \in I}$ is a finite family of closed subsets of $X$, whose interiors cover $X$ and $(D_i)_{i \in I}$ are $C^*$-algebras in $\mathcal{C}$, such that $\pi_{U_i}(\mathcal{F}) \subset \varepsilon/2 \varphi_i(\mathcal{H}_i)$ and such that for each direct summand of $D_i$ there is an element of $\mathcal{H}_i$ of norm $\geq \varepsilon$ which is contained and is full in that summand; the finite set $\mathcal{G}_i \subset D_i$ and $\delta_i > 0$ are given by Proposition 3.7 applied to the weakly semiprojective $C^*$-algebra $D_i$ for the input data $\mathcal{H}_i$ and $\varepsilon/2$; if $D_i$ is $KK$-stable, then $\mathcal{G}_i$ and $\delta_i$ are chosen such that the second part of Proposition 3.7 also applies.

**Lemma 4.5.** Let $A$ and $\mathcal{C}$ be as in Lemma 4.2. Suppose that each fiber of $A$ admits an exhaustive sequence of $C^*$-algebras isomorphic to $C^*$-algebras in $\mathcal{C}$. Then for any finite subset $\mathcal{F}$ of $A$ and any $\varepsilon > 0$ there is an $(\mathcal{F}, \varepsilon, \mathcal{C})$-approximation of $A$. Moreover, if $A$, $D$ and $\sigma$ are as in Lemma 4.3 and $\sigma_x \in KK(D, A(x))^{-1}$ for all $x \in X$, then there is an $(\mathcal{F}, \varepsilon, \mathcal{C})$-approximation of $A$ such that $\mathcal{C} = \{D\}$ and $KK(\varphi_i) = \sigma_{U_i}$ for all $i \in I$.

**Proof.** Since $X$ is compact, this is an immediate consequence of Lemmas 4.2, 4.3 and Proposition 3.7.

It is useful to consider the following operation of restriction. Suppose that $Y$ is a closed subspace of $X$ and let $(V_j)_{j \in J}$ be a finite family of closed subsets of $Y$ which refines the family $(Y \cap U_i)_{i \in I}$ and such that the interiors of the $V_j$’s form a cover of $Y$. Let $\iota : J \to I$ be a map such that $V_j \subseteq Y \cap U_{\iota(j)}$. Define

$${\iota^*}(\alpha) = \{\pi_Y(\mathcal{F}), \varepsilon, \{V_j, \pi_{V_j}, \varphi_{\iota(j)} : D_{\iota(j)} \to A(V_j), \mathcal{H}_{\iota(j)}, \mathcal{G}_{\iota(j)}, \delta_{\iota(j)}\}_{j \in J}\}.$$

It is obvious that $\iota^*(\alpha)$ is a $(\pi_Y(\mathcal{F}), \varepsilon, \mathcal{C})$-approximation of $A(Y)$. The operation $\alpha \mapsto \iota^*(\alpha)$ is useful even in the case $X = Y$. Indeed, by applying this procedure we can refine the cover of $X$ that appears in a given $(\mathcal{F}, \varepsilon, \mathcal{C})$-approximation of $A$.

An $(\mathcal{F}, \varepsilon, \mathcal{C})$-approximation $\alpha$ (as in (8)) is subordinated to an $(\mathcal{F}', \varepsilon', \mathcal{C})$-approximation, $\alpha' = \{\mathcal{F}', \varepsilon', \{U_i', \varphi_i' : D_i' \to A(U_i'), \mathcal{H}_i, \mathcal{G}_i, \delta_i\}_{i \in I}\}$, written $\alpha \prec \alpha'$, if

(i) $\mathcal{F} \subseteq \mathcal{F}'$,

(ii) $\varphi_i(\mathcal{G}_i) \subseteq \pi_{U_i}(\mathcal{F}')$ for all $i \in I$, and

(iii) $\varepsilon' < \min \{\varepsilon \cup \delta_i, i \in I\}$.

Let us note that, with notation as above, we have $\iota^*(\alpha) \prec \iota^*(\alpha')$ whenever $\alpha \prec \alpha'$.

The following theorem is the crucial technical result of our paper. It provides an approximation of continuous $C(X)$-algebras by subalgebras of category $\leq \dim(X)$.

**Theorem 4.6.** Let $\mathcal{C}$ be a class consisting of finite direct sums of weakly semiprojective simple $C^*$-algebras. Let $X$ be a finite dimensional compact metrizable space and let $A$ be a separable continuous $C(X)$-algebra the fibers of which admit exhaustive sequences of $C^*$-algebras isomorphic to $C^*$-algebras in $\mathcal{C}$. For any finite set $\mathcal{F} \subset A$ and any $\varepsilon > 0$ there exist $n \leq \dim(X)$ and an $n$-fibered $C$-monomorphism $(\psi_0, \ldots, \psi_n)$ into $A$ which induces a $*$-monomorphism $\eta : A(\psi_0, \ldots, \psi_n) \to A$ such that $\mathcal{F} \subset_{\varepsilon} \eta(A(\psi_0, \ldots, \psi_n))$. 
Proof. By Lemma 4.5, for any finite set $F \subset A$ and any $\varepsilon > 0$ there is an $(F, \varepsilon, C)$-approximation of $A$. Moreover, for any finite set $F \subset A$, any $\varepsilon > 0$ and any $n$, there is a sequence $\{\alpha_k : 0 \leq k \leq n\}$ of $(F_k, \varepsilon_k, C)$-approximations of $A$ such that $(F_0, \varepsilon_0) = (F, \varepsilon)$ and $\alpha_k$ is subordinated to $\alpha_{k+1}$:

$$\alpha_0 \prec \alpha_1 \prec \cdots \prec \alpha_n.$$

Indeed, assume that $\alpha_k$ was constructed. Let us choose a finite set $F_{k+1}$ which contains $F_k$ and liftings to $A$ of all the elements in $\bigcup_{i_k \in I_k} \varphi_{i_k}(G_{i_k})$. This choice takes care of the above conditions (i) and (ii). Next we choose $\varepsilon_{k+1}$ sufficiently small such that (iii) is satisfied. Let $\alpha_{k+1}$ be an $(F_{k+1}, \varepsilon_{k+1}, C)$-approximation of $A$ given by Lemma 4.5. Then obviously $\alpha_k \prec \alpha_{k+1}$. Fix a tower of approximations of $A$ as above where $n = \dim(X)$.

By [4, Lemma 3.2], for every open cover $\mathcal{V}$ of $X$ there is a finite open cover $\mathcal{U}$ which refines $\mathcal{V}$ and such that the set $\mathcal{U}$ can be partitioned into $n+1$ nonempty subsets consisting of elements with pairwise disjoint closures. Since we can refine simultaneously the covers that appear in a finite family $\{\alpha_k : 0 \leq k \leq n\}$ of approximations while preserving subordination, we may arrange not only that all $\alpha_k$ share the same cover $(U_i)_{i \in I}$, but moreover, that the cover $(U_i)_{i \in I}$ can be partitioned into $n+1$ subsets $U_0, \ldots, U_n$ consisting of mutually disjoint elements. For definiteness, let us write $U_k = \{U_{i_k} : i_k \in I_k\}$. Now for each $k$ we consider the closed subset of $X$

$$Y_k = \bigcup_{i_k \in I_k} U_{i_k},$$

the map $i_k : I_k \to I$ and the $(\pi_{Y_k}(F_k), \varepsilon_k, C)$-approximation of $A(Y_k)$, induced by $\alpha_k$, which is of the form

$$i_k(\alpha_k) = \pi_{Y_k}(F_k), \varepsilon, \{U_{i_k}, \varphi_{i_k} : D_{i_k} \to A(U_{i_k}), H_{i_k}, G_{i_k}, \delta_{i_k}\}_{i_k \in I_k},$$

where each $U_{i_k}$ is nonempty. We have

by construction. Since $\alpha_k \prec \alpha_{k+1}$ we obtain

$$F_k \subseteq F_{k+1},$$

for all $i_k \in I_k$,

$$\varepsilon_{k+1} < \min \left(\{\varepsilon_k\} \cup \{\delta_{i_k}, i_k \in I_k\}\right).$$

Set $X_k = Y_k \cup \cdots \cup Y_n$ and $E_k = \bigoplus_{i_k} C(U_{i_k}) \otimes D_{i_k}$ for $0 \leq k \leq n$. We shall construct a sequence of $C(Y_k)$-linear +-monomorphisms, $\psi_k : E_k \to A(Y_k)$, $k = n, \ldots, 0$, such that $(\psi_0, \ldots, \psi_n)$ is an $(n-k)$-fibered monomorphism into $A(X_k)$. Each map

$$\psi_k = \bigoplus_{i_k} \psi_{i_k} : E_k \to A(Y_k) = \bigoplus_{i_k} A(U_{i_k})$$

will have components $\psi_{i_k} : C(U_{i_k}) \otimes D_{i_k} \to A(U_{i_k})$ whose restrictions to $D_{i_k}$ will be perturbations of $\varphi_{i_k} : D_{i_k} \to A(U_{i_k})$, $i_k \in I_k$. We shall construct the maps $\psi_k$ by induction on decreasing $k$ such that if $B_k = A(X_k)(\psi_k, \ldots, \psi_n)$ and $\eta_k : B_k \to A(X_k)$ is the map induced by the $(n-k)$-fibered monomorphism $(\psi_k, \ldots, \psi_n)$, then

$$\pi_{X_{k+1} \cap U_{i_k}}(\psi_{i_k}(D_{i_k})) \subset \pi_{X_{k+1} \cap U_{i_k}}(\eta_{k+1}(B_{k+1})), \forall i_k \in I_k,$$
Note that (13) is equivalent to
\[ \pi \eta_k(B_k). \]

For the first step of induction, \( k = n \), we choose \( \psi_n = \oplus_i \tilde{\varphi}_{i,n} \) where \( \tilde{\varphi}_{i,n} : C(U_{i,n}) \otimes D_i \rightarrow A(U_{i,n}) \) are \( C(U_{i,n}) \)-linear extensions of the original \( \varphi_{i,n} \). Then \( B_n = E_n \) and \( \eta_n = \psi_n \). Assume that \( \psi_n, \ldots, \psi_{k+1} \) were constructed and that they have the desired properties. We shall construct now \( \psi_k \). Condition (14) formulated for \( k+1 \) becomes
\[ \pi_{X_{k+1}}(F_{k+1}) \subset C_{\varepsilon_{k+1}} \eta_{k+1}(B_{k+1}). \]
Since \( \varepsilon_{k+1} < \delta_{i,k} \), by using (11) and (16) we obtain
\[ \pi_{X_{k+1} \cap U_{i,k}}(\phi_{i,k}) \subset \delta_{i,k} \pi_{X_{k+1} \cap U_{i,k}}(\eta_{k+1}(B_{k+1})) \], for all \( i_k \in I_k \).

Conditions (9), (17) and (18) enable us to apply Lemma 4.4 and perturb \( \tilde{\varphi}_{i,k} \) to a \(*\)-monomorphism \( \psi_i : C(U_{i,k}) \otimes D_{i,k} \rightarrow A(U_{i,k}) \) satisfying (13) and (14) and such that
\[ KK(\psi_i|_{D_{i,k}}) = KK(\varphi_{i,k}) \]

if the algebras in \( C \) are assumed to be \( KK \)-stable. We set \( \psi_k = \oplus_i \psi_{i,k} \) and this completes the construction of \( \psi_0, \ldots, \psi_n \). Condition (14) for \( k = 0 \) gives \( F_0 = C_{\varepsilon}(B_0) = \eta(A(\psi_0, \ldots, \psi_n)) \). Thus \( \psi_0, \ldots, \psi_n \) satisfies the conclusion of the theorem. Finally let us note that it can happen that \( X_k = X \) for some \( k > 0 \). In this case \( F \subset C \psi_0, \ldots, \psi_n \) and for this reason we write \( n \leq \dim(X) \) in the statement of the theorem.

**Proposition 4.7.** Let \( X \) be a finite dimensional compact metrizable space and let \( A \) be a separable continuous \( C(X) \)-algebra the fibers of which are stable Kirchberg algebras. Let \( D \) be a \( KK \)-semiprojective stable Kirchberg algebra and suppose that there exists \( \sigma \in KK(D, A) \) such that \( \sigma_x \in KK(D, A(x))^{-1} \) for all \( x \in X \). For any finite subset \( F \) of \( A \) and any \( \varepsilon > 0 \) there is an \( n \)-fibered \( C \)-monomorphism \( (\psi_0, \ldots, \psi_n) \) into \( A \) such that \( n \leq \dim(X) \), \( C = \{ D \} \), and each component \( \psi_i : C(Y_i) \otimes D \rightarrow A(Y_i) \) satisfies \( KK(\psi_i) = \sigma_{Y_i} \), \( i = 0, \ldots, n \). Moreover, if \( \eta : A(\psi_0, \ldots, \psi_n) \rightarrow A \) is the induced \(*\)-monomorphism, then \( F \subset \varepsilon \eta(A(\psi_0, \ldots, \psi_n)) \) and \( KK(\eta_x) \) is a \( KK \)-equivalence for each \( x \in X \).

**Proof.** We repeat the proof of Theorem 4.6 while using only \( (\mathcal{F}_i, \varepsilon_i, \{ D \}) \)-approximations of \( A \) provided by the second part of Lemma 4.5. The outcome will be an \( n \)-fibered \( \{ D \} \)-monomorphism \( (\psi_0, \ldots, \psi_n) \) into \( A \) such that \( F \subset \varepsilon \eta(A(\psi_0, \ldots, \psi_n)) \). Moreover we can arrange that \( KK(\psi_i) = \sigma_{Y_i} \) for all \( i = 0, \ldots, n \), by (19), since \( KK(\varphi_{i,k}) = \sigma_U_{i,k} \) by Lemma 4.5. If \( x \in X \), and \( i = \min \{ k : x \in Y_k \} \), then \( \eta_x = (\psi_i)_x \), and hence \( KK(\eta_x) \) is a \( KK \)-equivalence.

**Remark 4.8.** Let us point out that we can strengthen the conclusion of Theorem 4.6 and Proposition 4.7 as follows. Fix a metric \( d \) for the topology of \( X \). Then we may arrange that there is a closed cover \( \{ Y_0, \ldots, Y_n \} \) of \( X \) and a number \( \ell > 0 \) such that \( \{ x : d(x, Y_i) \leq \ell \} \subset Y_i \) for \( i = 0, \ldots, n \).
Indeed, when we choose the finite closed cover \( \mathcal{U} = (U_i)_{i \in I} \) of \( X \) in the proof of Theorem 4.6 which can be partitioned into \( n + 1 \) subsets \( \mathcal{U}_0, \ldots, \mathcal{U}_n \) consisting of mutually disjoint elements, as given by [4, Lemma 3.2], and which refines all the covers \( \mathcal{U}(\alpha_0), \ldots, \mathcal{U}(\alpha_n) \) corresponding to \( \alpha_0, \ldots, \alpha_n \), we may assume that \( \mathcal{U} \) also refines the covers given by the interiors of the elements of \( \mathcal{U}(\alpha_0), \ldots, \mathcal{U}(\alpha_n) \). Since each \( U_i \) is compact and \( I \) is finite, there is \( \ell > 0 \) such that if \( V_i = \{ x : d(x, U_i) \leq \ell \} \), then the cover \( \mathcal{V} = (V_i)_{i \in I} \) still refines all of \( \mathcal{U}(\alpha_0), \ldots, \mathcal{U}(\alpha_n) \) and for each \( k = 0, \ldots, n \), the elements of \( \mathcal{V}_k = \{ V_i : U_i \in \mathcal{U}_k \} \), are still mutually disjoint. We shall use the cover \( \mathcal{V} \) rather than \( \mathcal{U} \) in the proof of the two theorems and observe that \( Y_k \) has the desired property. Finally let us note that if we define \( \psi'_i : E(Y'_i) \rightarrow A(Y'_i) \) by \( \psi'_i = \pi_{Y'_i} \psi_i \), then \( (\psi'_0, \ldots, \psi'_n) \) is an \( n \)-fibered \( \mathcal{C} \)-monomorphism into \( A \) which satisfies the conclusion of Theorem 4.6 and Proposition 4.7 since \( \pi_{Y'_i}(F) \subset \psi'_i(E_i) \) for all \( i = 0, \ldots, n \) and \( X = \bigcup_{i=1}^n Y'_i \).

5. Representing \( C(X) \)-algebras as inductive limits

We have seen that Theorem 4.6 yields exhaustive sequences for certain \( C(X) \)-algebras. In this section we show how to pass from an exhaustive sequence to a nested exhaustive sequence using semiprojectivity. The remainder of the paper does not depend on this section.

**Proposition 5.1.** Let \( X, A \) and \( C \) be as in Theorem 4.6. Let \( (\psi_0, \ldots, \psi_n) \) be an \( n \)-fibered \( \mathcal{C} \)-monomorphism into \( A \) with components \( \psi_i : E_i \rightarrow A(Y_i) \). Let \( \mathcal{F}_i \subset E_i, \mathcal{F} \subset A(\psi_0, \ldots, \psi_n) \) be finite sets and let \( \varepsilon > 0 \). Then there are finite sets \( G_i \subset E_i \) and \( \delta_i \geq 0, i = 0, \ldots, n \), such that for any \( C(X) \)-subalgebra \( A' \subset A \) which satisfies \( \psi_i(G_i) \subset \delta_i A(Y_i) \), \( i = 0, \ldots, n \), there is an \( n \)-fibered \( \mathcal{C} \)-monomorphism \( (\psi'_0, \ldots, \psi'_n) \) into \( A' \), with \( \psi'_i : E_i \rightarrow A'(Y_i) \) and such that (i) \( \| \psi_i(a) - \psi'_i(a) \| < \varepsilon \) for all \( a \in \mathcal{F}_i \) and all \( i \in \{0, \ldots, n\} \), (ii) \( (\psi_j)^{-1}_x(\psi_i)_x = (\psi'^{-1}_j)_x(\psi'_i)_x \) for all \( x \in Y_i \cap Y_j \) and \( 0 \leq i \leq j \leq n \). Moreover \( A(\psi_0, \ldots, \psi_n) = A'(\psi'_0, \ldots, \psi'_n) \) and the maps \( \eta : A(\psi_0, \ldots, \psi_n) \rightarrow A \) and \( \eta' : A'(\psi'_0, \ldots, \psi'_n) \rightarrow A' \) induced by \( (\psi_0, \ldots, \psi_n) \) and \( (\psi'_0, \ldots, \psi'_n) \) satisfy (iii) \( \| \eta(a) - \eta'(a) \| < \varepsilon \) for all \( a \in \mathcal{F} \).

**Proof.** Let us observe that if we prove (i) and (ii) then (iii) will follow by enlarging the sets \( \mathcal{F}_i \) so that \( p_i(F) \subset F_i \), where \( p_i : A(\psi_0, \ldots, \psi_n) \rightarrow E_i \) are the coordinate maps. We proceed now with the proof of (i) and (ii) by making some simplifications. We may assume that \( E_0 = C(Y_0) \otimes D_0 \) with \( D_0 \in C \) since the perturbations corresponding to disjoint closed sets can be done independently of each other. Without any loss of generality, we may assume that \( \mathcal{F}_0 \subset D_0 \) since we are working with morphisms on \( E_0 \) which are \( C(Y_0) \)-linear. We also enlarge \( \mathcal{F}_0 \) so that for each direct summand \( C \) of \( D_0 \), \( \mathcal{F}_0 \) contains an element \( c \) which is full in \( C \) and such that \( \| c \| \geq 2\varepsilon \).

The proof is by induction on \( n \). If \( n = 0 \) the statement follows from Proposition 3.7 and Lemma 4.1. Assume now that the statement is true for \( n - 1 \). Let \( E_i, \psi_i, A, A', \mathcal{F}_i, 1 \leq i \leq n \) and \( \varepsilon \) be as in the statement. For \( 0 \leq i < j \leq n \) let \( \eta_{j,i} : E_i(Y_i \cap Y_j) \rightarrow E_j(Y_i \cap Y_j) \) be the *-homomorphism of \( C(Y_i \cap Y_j) \)-algebras defined fiberwise by \( (\eta_{j,i})_x = (\psi^{-1}_j)_x(\psi_i)_x \).

Let \( G_0 \) and \( \delta_0 \) be given by Proposition 3.8 applied to the \( C^* \)-algebra \( D_0 \) for the input data \( \mathcal{F}_0 \) and \( \varepsilon \). For each \( 1 \leq j < n \) choose a finite subset \( \mathcal{H}_j \) of \( E_j \) whose restriction to \( Y_j \cap Y_0 \) contains \( \eta_{j,0}(G_0) \). Consider the sets \( \mathcal{F}'_j := \mathcal{F}_j \cup \mathcal{H}_j, 1 \leq j \leq n \) and the number \( \varepsilon' = \min\{ \delta_0, \varepsilon \} \). Let \( 1 \leq j \leq n \) and \( \mathcal{G}_j \) by given by the inductive assumption for \( n - 1 \) applied to \( A(X_1), A'(X_1), \psi_j, \mathcal{F}'_j \), where \( X_1 = Y_1 \cup \cdots \cup Y_n \).
We need to show that \( G_0, G_1, \ldots, G_n \) and \( \delta_0, \delta_1, \ldots, \delta_n \) satisfy the statement. By the inductive step there exists an \((n-1)\)-fibered \( \mathcal{C} \)-monomorphism \((\psi'_1, \ldots, \psi'_n)\) into \( A'(X_1) \) with components \( \psi_j' : E_j \to A'(Y_j) \) such that

(a) \( \| \psi_j(a) - \psi'_j(a) \| < \epsilon' = \min\{\delta_0, \epsilon\} \) for all \( a \in F_j \cup H_j \) and all \( 1 \leq j \leq n \),

(b) \( \psi_j^{-1}(\psi'_j)(x) = \psi'_j^{-1}(\psi'_j)(x) \) for all \( x \in Y_i \cap Y_j \) and \( 1 \leq i \leq j \leq n \).

The condition (b) enables to define a *-homomorphism \( \varphi : E_0 \to A'(Y_0 \cap X_1) \) with fiber maps \( \varphi_x = (\psi'_j)_{x}(\psi^{-1}_j)(\psi_0)_{x} \) for \( x \in Y_0 \cap Y_1 \) and \( 1 \leq j \leq n \).

Let us observe that \( \psi_0 : E_0 \to A(Y_0) \) is an approximate lifting of \( \varphi \). More precisely we have
\[
\| \pi_{X_1 \cap Y_0} \psi_0(a) - \varphi(a) \| < \delta_0 \text{ for all } a \in G_0.
\]
Indeed, for \( x \in Y_0 \cap Y_1 \), \( 1 \leq j \leq n \) and \( a \in G_0 \) we have
\[
\| (\psi_0)_x(a(x)) - (\psi'_j)_x(\psi^{-1}_j)(\psi_0)_x(a(x)) \| = \| (\psi_j)_x(\eta_j,0)_x(a(x)) - (\psi'_j)_x(\eta_j,0)_x(a(x)) \|
\leq \sup_{h \in H_j} \| \psi_j(h) - \psi'_j(h) \| < \epsilon' \leq \delta_0.
\]

Since we also have \( \psi_0(G_0) \subset \delta_0 A'(Y_0) \) by hypothesis, it follows from Proposition 3.8 that there exists \( \psi'_0 : D_0 \to A(Y_0) \) such that \( \| \psi'_0(a) - \psi_0(a) \| < \epsilon \) for all \( a \in F_0 \) and \( \pi_{Y_0 \cap X_1}^0 \psi'_0 = \varphi \). By Lemma 4.1 each \( \psi'_0 \) is injective since each \( (\psi_0)_x \) is injective. The \( C(Y_0) \)-linear extension of \( \psi'_0 \) to \( E_0 \) satisfies \( (\psi'_j)_x^{-1}(\psi_0)_x = (\psi'_j)_x^{-1}(\psi'_0)_x \) for all \( x \in Y_0 \cap Y_1 \) and \( 1 \leq j \leq n \) and this completes the proof of (ii). Condition (i) follows from (b).

The following result gives an inductive limit representation for continuous \( C(X) \)-algebras whose fibers are inductive limits of finite direct sums of simple semiprojective \( C^* \)-algebras. For example the fibers can be arbitrary AF-algebras or Kirchberg algebras which satisfy the UCT and whose \( K_1 \)-groups are torsion free. Indeed, by [29, Prop. 8.4.13], these algebras are isomorphic to inductive limits of sequences of Kirchberg algebras \( (D_n) \) with finitely generated K-theory groups and torsion free \( K_1 \)-groups. The algebras \( D_n \) are semiprojective by [33].

**Theorem 5.2.** Let \( \mathcal{C} \) be a class consisting of finite direct sums of semiprojective simple \( C^* \)-algebras. Let \( X \) be a finite dimensional compact metrizable space and let \( A \) be a separable continuous \( C(X) \)-algebra such that all its fibers admit exhaustive sequences consisting of \( C^* \)-algebras isomorphic to \( \mathcal{C} \)-algebras in \( \mathcal{C} \). Then \( A \) is isomorphic to the inductive limit of a sequence of continuous \( C(X) \)-algebras \( A_k \) such that \( \text{cat}_c(A_k) \leq \text{dim}(X) \).

**Proof.** By Theorem 4.6 and Proposition 5.1 we find a sequence \((\psi_0^{(k)}, \psi_n^{(k)})\) of \( n \)-fibered \( \mathcal{C} \)-monomorphisms into \( A \) which induces *-monomorphisms \( \eta^{(k)} : A_k = \psi_k^{(k)} \to A \) with the following properties. There is a sequence of finite sets \( F_k \subset A_k \) and a sequence of \( C(X) \)-linear *-monomorphisms \( \mu_k : A_k \to A_{k+1} \) such that

(i) \( \| \eta^{(k+1)} \mu_k(a) - \eta^{(k)}(a) \| < 2^{-k} \) for all \( a \in F_k \) and all \( k \geq 1 \),

(ii) \( \mu_k(F_k) \subset F_{k+1} \) for all \( k \geq 1 \),

(iii) \( \bigcup_{j=k}^{\infty} \mu_{k-1} \circ \cdots \circ \mu_k^{-1}(F_j) \) is dense in \( A_k \) and \( \bigcup_{j=k}^{\infty} \eta^{(j)}(F_j) \) is dense in \( A \) for all \( k \geq 1 \).

Arguing as in the proof of [29, Prop. 2.3.2], one verifies that
\[
\varphi_k(a) = \lim_{j \to \infty} \eta^{(j)} \circ (\mu_{j-1} \circ \cdots \circ \mu_k)(a)
\]
defines a sequence of *-monomorphisms \( \varphi_k : A_k \to A \) such that \( \varphi_{k+1} \mu_k = \varphi_k \) and the induced map \( \varphi : \lim_{k \to \infty} (A_k, \mu_k) \to A \) is an isomorphism of \( C(X) \)-algebras. \( \square \)
Remark 5.3. By similar arguments one proves a unital version of Theorem 5.2.

6. When is a fibered product locally trivial

For C*-algebras $A$, $B$ we endow the space $\text{Hom}(A, B)$ of *-homomorphisms with the point-norm topology. If $X$ is a compact Hausdorff space, then $\text{Hom}(A, C(X) \otimes B)$ is homeomorphic to the space of continuous maps from $X$ to $\text{Hom}(A, B)$ endowed with the compact-open topology. We shall identify a *-homomorphism $\varphi \in \text{Hom}(A, C(X) \otimes B)$ with the corresponding continuous map $X \to \text{Hom}(A, B)$, $x \mapsto \varphi_x$, $\varphi_x(a) = \varphi(a)(x)$ for all $x \in X$ and $a \in A$. Let $D$ be a C*-algebra and let $A$ be a $C(X)$-algebra. If $\alpha : D \to A$ is a *-homomorphism, let us denote by $\tilde{\alpha} : C(X) \otimes D \to A$ its (unique) $C(X)$-linear extension and write $\tilde{\alpha} \in \text{Hom}_{C(X)}(C(X) \otimes D, A)$. For C*-algebras $D$, $B$ we shall make without further comment the following identifications

$$\text{Hom}_{C(X)}(C(X) \otimes D, C(X) \otimes B) \equiv \text{Hom}(D, C(X) \otimes B) \equiv C(X, \text{Hom}(D, B)).$$

For a C*-algebra $D$ we denote by $\text{End}(D)$ the set of full (and unital if $D$ is unital) *-endomorphisms of $D$ and by $\text{End}(D)^0$ the path component of $\text{id}_D$ in $\text{End}(D)$. Let us consider

$$\text{End}(D)^* = \{ \gamma \in \text{End}(D) : KK(\gamma) \in KK(D, D)^{-1} \}.$$  

Proposition 6.1. Let $X$ be a compact metrizable space and let $D$ be a $KK$-semiprojective Kirchberg algebra. Let $\alpha : D \to C(X) \otimes D$ be a full (and unital, if $D$ is unital) *-homomorphism such that $KK(\alpha_x) \in KK(D, D)^{-1}$ for all $x \in X$. Then there is a full *-homomorphism $\Phi : D \to C(X \times [0, 1]) \otimes D$ such that $\Phi(x, 0) = \alpha_x$ and $\Phi(x, t) \in \text{Aut}(D)$ for all $x \in X$ and $t \in (0, 1)$. Moreover, if $\Phi_1 : D \to C(X) \otimes D$ is defined by $\Phi_1(d)(x) = \Phi(x, 1)(d)$, for all $d \in D$ and $x \in X$, then $\alpha \approx_{ah} \Phi_1$.

Proof. Since $X$ is a metrizable compact space, $X$ is homeomorphic to the projective limit of a sequence of finite simplicial complexes $(X_i)$ by [13, Thm. 10.1, p.284]. Since $D$ is $KK$-semiprojective, $KK(D, \lim C(X_i) \otimes D) = KK(D, C(X) \otimes D)$ by Theorem 3.12. By Theorem 3.1, there is $i$ and a full (and unital if $D$ is unital) *-homomorphism $\varphi : D \to C(X_i) \otimes D$ whose $KK$-class maps to $KK(\alpha) \in KK(D, C(X) \otimes D)$. To summarize, we have found a finite simplicial complex $Y$, a continuous map $h : X \to Y$ and a continuous map $y \mapsto \varphi_y \in \text{End}(D)$, defined on $Y$, such that the full (and unital if $D$ is unital) *-homomorphism $h^* \varphi : D \to C(X) \otimes D$ corresponding to the continuous map $x \mapsto \varphi_{h(x)}$ satisfies $KK(h^* \varphi) = KK(\alpha)$. We may arrange that $h(X)$ intersects all the path components of $Y$ by dropping the path components which are not intersected. Since $\alpha_x \in \text{End}(D)^*$ by hypothesis, and since $KK(\alpha_x) = KK(\varphi_{h(x)})$, we infer that $\varphi_y \in \text{End}(D)^*$ for all $y \in Y$. We shall find a continuous map $y \mapsto \psi_y \in \text{End}(D)^*$ defined on $Y$, such that the maps $y \mapsto \psi_y \varphi_y$ and $y \mapsto \varphi_y \psi_y$ are homotopic to the constant map $\iota$ that takes $Y$ to $\text{id}_D$. It is clear that it suffices to deal separately with each path component of $Y$, so that for this part of the proof we may assume that $Y$ is connected. Fix a point $z \in Y$. By [29, Thm. 8.4.1] there is $\nu \in \text{Aut}(D)$ such that $KK(\nu^{-1}) = KK(\varphi_z)$ and hence $KK(\nu \varphi_z) = KK(\text{id}_D)$. By Theorem 3.1, there is a unitary $u \in M(D)$ such that $uw \varphi_z(-)u^*$ is homotopic to $\text{id}_D$. Let us set $\theta = uw(-)u^* \in \text{Aut}(D)$ and observe that $\theta \varphi_z \in \text{End}(D)^0$. Since $Y$ is path connected, it follows that the entire image of the map $y \mapsto \theta \varphi_y$ is contained in $\text{End}(D)^0$. Since $\text{End}(D)^0$ is a path connected H-space with unit element, it follows by [38, Thm. 2.4, p.462] that the homotopy classes $[Y, \text{End}(D)^0]$ (with no condition on basepoints, since the action of the fundamental group $\pi_1(\text{End}(D)^0, \text{id}_D)$ is trivial by [38,
3.6, p166) form a group under the natural multiplication. Therefore we find $y \mapsto \psi_y' \in \text{End}(D)^0$ such that $y \mapsto \psi_y' \theta \varphi_y$ and $y \mapsto \theta \varphi_y \psi_y'$ are homotopic to $\iota$. It follows that $y \mapsto \psi_y' \theta \equiv \psi_y' \theta$ is the homotopic inverse of $y \mapsto \varphi_y$ in $[Y, \text{End}(D)^*]$. Composing with $h$ we obtain that the maps $x \mapsto \varphi_{h(x)} \psi_{h(x)}$ are homotopic to the constant map that takes $X$ to $\text{id}_D$. By the homotopy invariance of $KK$-theory we obtain that
$$\text{KK}(\tilde{h}^* \varphi) = \text{KK}(\tilde{h}^* \psi \cdot \varphi) = \text{KK}(\iota_D),$$
where $\tilde{h}^* \varphi$ and $\tilde{h}^* \psi$ denote the $C(X)$-linear extensions of the corresponding maps and $\iota_D : D \to C(X) \otimes D$ is defined by $\iota_D(d) = 1_{C(X)} \otimes d$ for all $d \in D$. Let us recall that $\text{KK}(h^* \varphi) = \text{KK}(\alpha)$ and hence $\text{KK}(\tilde{h}^* \varphi) = \text{KK}(\tilde{\alpha})$. If we set $\Psi = h^* \psi$, then
$$\text{KK}(\tilde{\alpha} \Psi) = \text{KK}(\tilde{\Psi} \alpha) = \text{KK}(\iota_D).$$
By Theorem 3.1 $\tilde{\alpha} \Psi \approx_{u} \iota_D$ and $\tilde{\Psi} \alpha \approx_{u} \iota_D$, and hence $\tilde{\alpha} \tilde{\Psi} \approx_{u} \text{id}_{C(X) \otimes D}$ and $\tilde{\Psi} \tilde{\alpha} \approx_{u} \text{id}_{C(X) \otimes D}$. By [29, Cor. 2.3.4], there is an isomorphism $\Gamma : C(X) \otimes D \to C(X) \otimes D$ such that $\Gamma \approx_{u} \tilde{\alpha}$. In particular $\Gamma$ is $C(X)$-linear and $\Gamma_x \in \text{Aut}(D)$ for all $x \in X$. Replacing $\Gamma$ by $u \Gamma(\cdot) u^*$ for some unitary $u \in M(C(X) \otimes D)$ we can arrange that $\Gamma|_D$ is arbitrarily close to $\alpha$. Therefore $\text{KK}((\Gamma|_D) = \text{KK}(\alpha)$ since $D$ is KK-stable. By Theorem 3.1 there is a continuous map $[0, 1] \to U(M(C(X) \otimes D))$, $t \mapsto u_t$, with the property that
$$\lim_{t \to 0} \|u_t \Gamma(a) u_t^* - \alpha(a)\| = 0,$$
for all $a \in D$. Therefore the equation
$$\Phi(x, t) = \begin{cases} \alpha_x, & \text{if } t = 0, \\ u_t(x) \Gamma_x u_t(x)^*, & \text{if } t \in (0, 1], \end{cases}$$
defines a continuous map $\Phi : X \times [0, 1] \to \text{End}(D)^*$ which extends $\alpha$ and such that $\Phi(X \times (0, 1]) \subset \text{Aut}(D)$. Since $\alpha$ is homotopic to $\Phi_1$, we have that $\alpha \approx_{uh} \Phi_1$ by Theorem 3.1.

**Proposition 6.2.** Let $X$ be a compact metrizable space and let $D$ be a KK-semiprojective Kirchberg algebra. Let $Y$ be a closed subset of $X$. Assume that a map $\gamma : Y \to \text{End}(D)^*$ extends to a continuous map $\alpha : X \to \text{End}(D)^*$. Then there is a continuous extension $\eta : X \to \text{End}(D)^*$ of $\gamma$, such that $\eta(X \setminus Y) \subset \text{Aut}(D)$.

**Proof.** Since the map $x \mapsto \alpha_x$ takes values in $\text{End}(D)^*$, by Proposition 6.1 there exists a continuous map $\Phi : X \times [0, 1] \to \text{End}(D)^*$ which extends $\alpha$ and such that $\Phi(X \times (0, 1]) \subset \text{Aut}(D)$. Let $d$ be a metric for the topology of $X$ such that $\text{diam}(X) \leq 1$. The equation $\eta(x) = \Phi(x, d(x, Y))$ defines a map on $X$ that satisfies the conclusion of the proposition.

**Lemma 6.3.** Let $X$ be a compact metrizable space and let $D$ be a KK-semiprojective Kirchberg algebra. Let $Y$ be a closed subset of $X$. Let $\alpha : Y \times [0, 1] \cup X \times \{0\} \to \text{End}(D)$ be a continuous map such that $\alpha(x, 0) \in \text{End}(D)^*$ for all $x \in X$. Suppose that there is an open set $V$ in $X$ which contains $Y$ and such that $\alpha$ extends to a continuous map $\alpha_V : V \times [0, 1] \cup X \times \{0\} \to \text{End}(D)$. Then there is $\eta : X \times [0, 1] \to \text{End}(D)^*$ such that $\eta$ extends $\alpha$ and $\eta(x, t) \in \text{Aut}(D)$ for all $x \in X \setminus Y$ and $t \in (0, 1]$.

**Proof.** By Proposition 6.2 it suffices to find a continuous map $\tilde{\alpha} : X \times [0, 1] \to \text{End}(D)^*$ which extends $\alpha$. Fix a metric $d$ for the topology of $X$ and define $\lambda : X \to [0, 1]$ by $\lambda(x) = d(x, X \setminus Y)$.
Proposition 6.4. Let $X$ be a compact metrizable space and let $D$ be a $KK$-semiprojective stable Kirchberg algebra. Let $A$ be a separable $C(X)$-algebra which is locally isomorphic to $C(X) \otimes D$. Suppose that there is an isomorphism of $C(X)$-algebras $\psi : C(X) \otimes D \to A$ such that $\text{ker}(\psi) = \sigma$.

Proof. Since $X$ is compact and $A$ is locally trivial it follows that $\text{cat}_D(A) = \infty$. By Lemma 2.9, $A \cong pA \otimes \mathbb{C} \otimes K$ for some projection $p \in A$. By Theorem 2.1, there is a full $*$-homomorphism $\varphi : D \to A$ such that $K K(D, A) = \sigma$. We shall construct an isomorphism of $C(X)$-algebras $\psi : C(X) \otimes D \to A$ such that $\psi$ is homotopic to $\tilde{\varphi}$, the $C(X)$-linear extension of $\varphi$. Moreover the homotopy $(H_t)_{t \in [0,1]}$ will have the property that $H(t, x, t) : D \to A(x)$ is an isomorphism for all $x \in X$ and $t > 0$. We prove this by induction on numbers $n$ with the property that there are two closed covers of $X$, $W_1, \ldots, W_n$ and $Y_1, \ldots, Y_n$ such that $Y_i$ is contained in the interior of $W_i$ and $A(W_i) \cong C(Y_i) \otimes D$ for $1 \leq i \leq n$. First we observe that the case $n = 1$ follows from Proposition 6.2. Let us now pass from $n - 1$ to $n$. Given two covers as above, there is yet another closed cover $V_1, \ldots, V_n$ of $X$ such that $V_i$ is a neighborhood of $Y_i$ and $W_i$ is a neighborhood of $V_i$ for all $1 \leq i \leq n$. Set $Y = \bigcup_{i=1}^{n-1} Y_i$, $V = \bigcup_{i=1}^{n-1} V_i$ and $W = \bigcup_{i=1}^{n-1} W_i$. By the inductive hypothesis applied to $A(V)$, and the covers $V_1, \ldots, V_n-1$ and $W_1 \cap V, \ldots, W_n-1 \cap V$ there is a homomorphism $h : D \to A(V) \otimes C[0,1] \otimes C[0,1] \otimes \mathbb{C}$ such that $h((x, t)) = \varphi_x$ and $h((x, t)) : D \to A(x)$ is an isomorphism for all $(x, t) \in V \times [0,1]$. Fix a trivialization $\nu : A(Y_{n+1}) \to C(Y_{n+1}) \otimes D$. Define a continuous map $\alpha : (V \cap Y_{n+1}) \times [0,1] \cup Y_{n+1} \times \{0\} \to \text{End}(D)$ by setting $\alpha((x, t)) = \nu_x h((x, t))$ if $(x, t) \in (V \cap Y_{n+1}) \times [0,1]$ and $\alpha((x, 0)) = \nu_x \varphi_x$ if $(x, 0) \in Y_{n+1}$. Since $V \cap Y_{n+1}$ is a neighborhood of $Y \cap Y_{n+1}$ in $Y_{n+1}$ and since $\nu_x \varphi_x \in \text{End}(D)^*$ for all $x \in Y_{n+1}$, by Lemma 6.3 there is a continuous map $\eta : Y_{n+1} \times [0,1] \to \text{End}(D)^*$ which extends the restriction of $\alpha$ to $(Y \cap Y_{n+1}) \times [0,1] \cup Y_{n+1} \times \{0\}$. We conclude the construction of the desired homotopy by defining $H : D \to A(X) \otimes C[0,1]$ by $H(x, t) = h((x, t))$ for $(x, t) \in V \times [0,1]$ and $H(x, t) = \nu_x^{-1} \eta(x, t)$ for $(x, t) \in Y_{n+1} \times [0,1]$. 

Lemma 6.5. Let $D$ be a $KK$-semiprojective stable Kirchberg algebra. Let $X$ be a compact metrizable space and $Y, Z$ be closed subsets of $X$ such that $X = Y \cup Z$. Suppose that $\gamma : D \to C(Y \cap Z) \otimes D$ is a full $*$-homomorphism which admits a lifting to a full $*$-homomorphism $\alpha : D \to C(Y) \otimes D$ such that $\alpha_x \in \text{End}(D)^*$ for all $x \in Y$. Then the pullback $C(Y) \otimes D \oplus_{\pi_{Y \cap Z} \otimes \pi_{Y \cap Z}} C(Z) \otimes D$ is isomorphic to $C(X) \otimes D$.

Proof. By Prop. 6.2 there is a $*$-homomorphism $\eta : D \to C(Y) \otimes D$ such that $\eta_x = \gamma_x$ for $x \in Y \cap Z$ and such that $\eta_x \in \text{Aut}(D)$ for $x \in Y \setminus Z$. Using the short five lemma one checks immediately that the triple $(\eta \otimes \pi_{Y \cap Z}, \text{id}_{C(Z) \otimes D})$ defines a $C(X)$-linear isomorphism:

$$
C(X) \otimes D = C(Y) \otimes D \oplus_{\pi_{Y \cap Z} \otimes \pi_{Y \cap Z}} C(Z) \otimes D \to C(Y) \otimes D \oplus_{\pi_{Y \cap Z} \otimes \pi_{Y \cap Z}} C(Z) \otimes D
$$

Lemma 6.6. Let $D$ be a $KK$-semiprojective stable Kirchberg algebra. Let $Y, Z$ and $Z'$ be closed subsets of a compact metrizable space $X$ such that $Z'$ is a neighborhood of $Z$ and $X = Y \cup Z$. Let $B$ be a $C(Y)$-algebra locally isomorphic to $C(Y) \otimes D$ and let $E$ be a $C(Z')$-algebra locally isomorphic to $C(Z') \otimes D$. Let $\alpha : E(Y \cap Z') \to B(Y \cap Z')$ be a $*$-monomorphism of $C(Y \cap Z')$-algebras such
that \( KK(\alpha_x) \in KK(E(x), B(x))^{-1} \) for all \( x \in Y \cap Z' \). If \( \gamma = \alpha_{Y \cap Z} \), then \( B(Y) \oplus \pi_{Y \cap Z} \gamma \pi_{Y \cap Z} E(Z) \) is locally isomorphic to \( C(X) \otimes D \).

**Proof.** Since we are dealing with a local property, we may assume that \( B = C(Y) \otimes D \) and \( E = C(Z') \otimes D \). To simplify notation we let \( \pi \) stand for both \( \pi_{Y \cap Z} \) and \( \pi_{Z' \cap Z} \) in the sequel. Let us denote by \( H \) the \( C(X) \)-algebra \( C(Y) \otimes D \oplus \pi, \gamma \pi C(Z) \otimes D \). We must show that \( H \) is locally trivial. Let \( x \in X \). If \( x \notin Z \), then there is a closed neighborhood \( V \) of \( x \) which does not intersect \( Z \), and hence the restriction of \( H \) to \( V \) is isomorphic to \( C(V) \otimes D \), as it follows immediately from the definition of \( H \). It remains to consider the case when \( x \in Z \). Now \( Z' \) is a closed neighborhood of \( x \) in \( X \) and the restriction of \( H \) to \( Z' \) is isomorphic to \( C(Y \cap Z') \otimes D \oplus \pi, \gamma \pi C(Z) \otimes D \). Since \( \gamma : Y \cap Z \to \text{End}(D)^* \) admits a continuous extension \( \alpha : Y \cap Z' \to \text{End}(D)^* \), it follows that \( H(Z') \) is isomorphic to \( C(Z') \otimes D \) by Lemma 6.5. \( \square \)

**Proposition 6.7.** Let \( X, A, D \) and \( \sigma \) be as in Proposition 4.7. For any finite subset \( F \) of \( A \) and any \( \varepsilon > 0 \) there is a \( C(X) \)-algebra \( B \) which is locally isomorphic to \( C(X) \otimes D \) and there exists a \( C(X) \)-linear \( * \)-monomorphism \( \eta : B \to A \) such that \( F \subseteq \eta(B) \) and \( KK(\eta_x) \in KK(B(x), A(x))^{-1} \) for all \( x \in X \).

**Proof.** Let \( \psi_k : E_k = C(Y_k) \otimes D \to A(Y_k), \ k = 0, ..., n \) be as in the conclusion of Proposition 4.7, strengthened as in Remark 4.8. Therefore we may assume that there is another \( n \)-fibered \( \{D\} \)-monomorphism \((\psi_0', ..., \psi_n')\) into \( A \) such that \( \psi_k : C(Y'_k) \otimes D \to A(Y'_k), \ Y'_k \) is a closed neighborhood of \( Y_k \), and \( \pi_{Y_k} \psi'_k = \psi_k, k = 0, ..., n \). Let \( X_k, B_k, \eta_k \) and \( \gamma_k \) be as in Definition 2.8. \( B_0 \) and \( \eta_0 \) satisfy the conclusion of the proposition, except that we need to prove that \( B_0 \) is locally isomorphic to \( C(X) \otimes D \). We prove by induction on decreasing \( k \) that the \( C(X_k) \)-algebras \( B_k \) are locally trivial. Indeed \( B_n = C(X_n) \otimes D \) and assuming that \( B_k \) is locally trivial, it follows by Lemma 6.6 that \( B_{k-1} \) is locally trivial, since by (5)

\[
B_{k-1} \cong B_k \oplus \pi_{\eta_k \pi \psi_{k-1} E_{k-1} \oplus \pi_{\gamma_k \pi} E_{k-1}}, \quad (\pi = \pi_{Y_k \cap Y_{k-1}})
\]

and\( \gamma : E_{k-1}(X_k \cap Y_k, Y_{k-1} \cap Y_{k-1}) \to B_k(X_k \cap Y_{k-1}) \), \( \gamma_{k-1} = (\eta_k)^{-1}(\psi_{k-1}) \), extends to a \( * \)-monomorphism \( \alpha : E_{k-1}(X_k \cap Y_{k-1}, Y_{k-1} \cap Y_{k-1}) \to B_k(X_k \cap Y_{k-1}) \), \( \alpha_x = (\eta_k)^{-1}(\psi_{k-1}) \) and \( KK(\alpha_x) \) is a KK-equivalence since both \( KK((\eta_k)_x) \) and \( KK((\psi_{k-1})_x) \) are KK-equivalences. \( \square \)

**7. WHEN IS A C(X)-ALGEBRA LOCALLY TRIVIAL**

In this section we prove Theorems 1.1 – 1.5 and some of their consequences.

**Proof of Theorem 1.2.**

**Proof.** Let \( X \) denote the primitive spectrum of \( A \). Then \( A \) is a continuous \( C(X) \)-algebra and its fibers are stable Kirchberg algebras (see [5, 2.2.2]). Since \( A \) is separable, \( X \) is metrizable by Lemma 2.2. By Proposition 6.7 there is a sequence of \( C(X) \)-algebras \( (A_k)_{k=1}^{\infty} \) locally isomorphic to \( C(X) \otimes D \) and a sequence of \( C(X) \)-linear \( * \)-monomorphisms \( (\eta_k : A_k \to A)_{k=1}^{\infty} \), such that \( KK(\eta_k)_x \) is a KK-equivalence for each \( x \in X \) and \( (\eta_k(A_k))_{k=1}^{\infty} \) is an exhaustive sequence of \( C(X) \)-subalgebras of \( A \). Since \( D \) is weakly semiprojective and KK-stable, after passing to a subsequence of \( (A_k) \) if necessary, we find a sequence \( (\sigma_k)_{k=1}^{\infty}, \sigma_k \in KK(D, A_k) \) such that \( KK(\eta_k)_x \sigma_k = \sigma \) for all \( k \geq 1 \). Since both \( KK(\eta_k)_x \) and \( \sigma_x \) are KK-equivalences, we deduce that \( (\sigma_k)_x \in KK(D, A_k(x))^{-1} \) for all \( x \in X \). By Proposition 6.4, for each \( k \geq 1 \) there is an isomorphism of \( C(X) \)-algebras
\( \varphi_k : C(X) \otimes D \to A_k \) such that \( KK(\varphi_k) = \sigma_k \). Therefore if we set \( \theta_k = \eta_k \varphi_k \), then \( \theta_k \) is a \( C(X) \)-linear \(*\)-monomorphism from \( B \) to \( A \) such that \( KK(\theta_k) = \sigma \) and \( \theta_k(B) \) is isomorphic to \( B \) this property implies that each map \( \mu_k \) is isomorphism of \( B \), that we can arrange that \( \psi \), such that \( \psi(1) = 1 \) as well. Then the continuous \( C[0,1] \)-algebra \( E = \{ f \in C[0,1] \otimes D : f(0) \in \psi(D) \} \) is locally trivial if and only if \( \psi \) is homotopic to an automorphism of \( D \).

**Proof.** Suppose that \( E \) is trivial on some neighborhood of 0. Thus there is \( s \in (0,1] \) such that \( C[0,s] \otimes D \cong E[0,s] \). Since \( E[0,s] \subset C[0,s] \otimes D \), there is a continuous path \( (\theta_t)_{t \in [0,s]} \) in \( \text{End}(D) \) such that \( \theta_0(D) = \psi(D) \). Set \( \beta = \psi^{-1} \theta_0 \). Then \( \psi \) is homotopic to an automorphism via the path \( (\theta_t)_{t \in [0,s]} \). Conversely, if \( \psi \) is homotopic to an automorphism \( \alpha \), then by Theorem 3.1 there is a continuous path \( (\theta_t)_{t \in [0,1]} \) of unitaries in \( D^+ \) such that \( \lim_{t \to 0} \| \psi(d) - u_t \alpha(d) u_t^* \| = 0 \) for all \( d \in D \). The path \( (\theta_t)_{t \in [0,1]} \) defined by \( \theta_0 = \psi \) and \( \theta_t = u_t \alpha u_t^* \) for \( t \in [0,1] \) induces a \( C[0,1] \)-linear \(*\)-endomorphism of \( C[0,1] \otimes D \) which maps injectively \( C[0,1] \otimes D \) onto \( E \).

**Proof of Theorem 1.3.**

For the first part we apply Theorem 1.2 for \( D = O_2 \otimes K \) and \( \sigma = 0 \). For the second part we assert that if \( D \) is a Kirchberg such that all continuous \( C[0,1] \)-algebras with fibers isomorphic to \( D \) are locally trivial then \( D \) is stable and \( KK(D,D) = 0 \). This implies that \( D \) is KK-equivalent.
to \( \mathcal{O}_2 \) and hence that \( D \cong \mathcal{O}_2 \otimes K \) by [29, Thm. 8.4.1]. The Kirchberg algebra \( D \) is either unital or stable [29, Prop. 4.1.3]. Let \( \psi : D \to D \) be a \( * \)-monomorphism such that \( KK(\psi) = 0 \) and such that \( \psi(1_D) < 1_D \) if \( D \) is unital. By Proposition 7.1 \( \psi \) is homotopic to an automorphism of \( \theta \) of \( D \). Therefore \( D \) must be nonunital (and hence stable), since otherwise \( 1_D \) would be homotopic to its proper subprojection \( \psi(1_D) \). Moreover \( KK(\theta) = KK(\psi) = 0 \) and hence \( KK(D, D) = 0 \) since \( \theta \) is an automorphism. 

Dixmier and Douady [12] proved that a continuous field with fibers \( K \) over a finite dimensional locally compact Hausdorff space is locally trivial if and only it verifies Fell’s condition, i.e. for each \( x_0 \in X \) there is a continuous section \( a \) of the field such that \( a(x) \) is a rank one projection for each \( x \) in a neighborhood of \( x_0 \). We have a analogous result:

**Corollary 7.2.** Let \( A \) be a separable \( C^* \)-algebra whose primitive spectrum \( X \) is Hausdorff and of finite dimension. Suppose that for each \( x \in X \), \( A(x) \) is KK-semiprojective, nuclear, purely infinite and stable. Then \( A \) is locally trivial if and only if for each \( x \in X \) there exist a closed neighborhood \( V \) of \( x \), a Kirchberg algebra \( D \) and \( \sigma \in KK(D, A(V)) \) such that \( \sigma_v \in KK(D, A(v))^{-1} \) for each \( v \in V \).

**Proof.** One applies Theorem 1.2 for \( D \otimes K \) and \( A(V) \).

We turn now to unital \( C(X) \)-algebras.

**Theorem 7.3.** Let \( A \) be a separable unital \( C(X) \)-algebra over a finite dimensional compact Hausdorff space \( X \). Suppose that each fiber \( A(x) \) is nuclear simple and purely infinite. Then \( A \) is isomorphic to \( C(X) \otimes D \), for some KK-semiprojective unital Kirchberg algebra \( D \), if and only if there is \( \sigma \in KK(D, A) \) such that \( K_0(\sigma)[1_D] = [1_A] \) and \( \sigma_x \in KK(D, A(x))^{-1} \) for all \( x \in X \). For any such \( \sigma \) there is an isomorphism of \( C(X) \)-algebras \( \Phi : C(X) \otimes D \to A \) such that \( KK(\Phi|_D) = \sigma \).

**Proof.** We verify the nontrivial implication. \( X \) is metrizable by Lemma 2.2. \( A \) is a continuous \( C(X) \)-algebra by Lemma 2.3. By Theorem 1.2, there is an isomorphism \( \Phi : C(X) \otimes D \otimes K \to A \otimes K \) such that \( KK(\Phi) = \sigma \). Since \( K_0(\sigma)[1_D] = [1_A] \) and since \( A \otimes K \) contains a full properly infinite projection, we may arrange that \( \Phi(1_{C(X) \otimes D} \otimes e_{11}) = 1_A \otimes e_{11} \) after conjugating \( \Phi \) by some unitary \( u \in M(A \otimes K) \). Then \( \varphi = \Phi|_{C(X) \otimes D \otimes e_{11}} \) satisfies the conclusion of the theorem. 

**Proof of Theorem 1.4.**

**Proof.** Let \( D \) be a KK-semiprojective unital Kirchberg algebra \( D \) such that every unital \( * \)-endomorphism of \( D \) is a KK-equivalence. Suppose that \( A \) is a separable unital \( C(X) \)-algebra over a finite dimensional compact Hausdorff space the fibers of which are isomorphic to \( D \). We shall prove that \( A \) is locally trivial. By Theorem 7.3, it suffices to show that each point \( x_0 \in X \) has a closed neighborhood \( V \) for which there is \( \sigma \in KK(D, A(V)) \) such that \( K_0(\sigma)[1_D] = [1_{A(V)}] \) and \( \sigma_x \in KK(D, A(x))^{-1} \) for all \( x \in V \).

Let \( (V_n)_{n=1}^\infty \) be a decreasing sequence of closed neighborhoods of \( x_0 \) whose intersection is \( \{x_0\} \). Then \( A(x_0) \cong \lim A(V_n) \). By assumption, there is an isomorphism \( \eta : D \to A(x_0) \). Since \( D \) is KK-semiprojective, there is \( m \geq 1 \) such that \( KK(\eta) \) lifts to some \( \sigma \in KK(D, A(V_m)) \) such that \( K_0(\sigma)[1_D] = [1_{A(V_m)}] \). Let \( x \in V_m \). By assumption, there is an isomorphism \( \varphi : A(x) \to D \). The \( K_0 \)-morphism induced by \( KK(\varphi) \sigma_x \) maps \( [1_D] \) to itself. By Theorem 3.1 there is a unital
\*homomorphism $\psi : D \to D$ such that $KK(\psi) = KK(\varphi)\sigma_x$. By assumption we must have $KK(\psi) \in KK(D, D)^{-1}$ and hence $\sigma_x \in KK(D, A(x))^{-1}$ since $\varphi$ is an isomorphism. Therefore $A(V_m) \cong C(V_m) \otimes D$ by Theorem 7.3.

Conversely, let us assume that all separable unital continuous $C[0, 1]$-algebras with fibers isomorphic to $D$ are locally trivial. Let $\psi$ be any unital \*endomorphism of $D$. By Proposition 7.1 $\psi$ is homotopic to an automorphism of $D$ and hence $KK(\psi)$ is invertible. \hfill \Box

**Proof of Theorem 1.1**

Proof. Let $A$ be as in Theorem 1.1 and let $n \in \{2, 3, \ldots\} \cup \{\infty\}$. It is known that $O_n$ satisfies the UCT. Moreover $K_0(O_n)$ is generated by $[1_{O_n}]$ and $K_1(O_n) = 0$. Therefore any unital \*endomorphism of $O_n$ is a KK-equivalence. It follows that $A$ is locally trivial by Theorem 1.4. Suppose now that $n = 2$. Since $KK(O_2, O_2) = KK(O_2, A) = 0$, we may apply Theorem 1.4 with $\sigma = 0$ and obtain that $A \cong C(X) \otimes O_2$. Suppose now that $n = \infty$. Let us define $\theta : K_0(O_\infty) \to K_0(A)$ by $\theta(k[1_{O_\infty}]) = k[1_A]$, $k \in \mathbb{Z}$. Since $O_\infty$ satisfies the UCT, $\theta$ lifts to some element $\sigma \in KK(O_\infty, A)$. By Theorem 1.4 it follows that $A \cong C(X) \otimes O_\infty$. Finally let us consider the case $n \in \{3, 4, \ldots\}$. Then $K_0(O_n) = \mathbb{Z}/(n-1)$. Since $O_n$ satisfies the UCT, the existence of an element $\sigma \in KK(O_n, A)$ such that $K_0(\sigma)[1_{O_n}] = [1_A]$ is equivalent to the existence of a morphism of groups $\theta : \mathbb{Z}/(n-1) \to K_0(A)$ such that $\theta(1) = [1_A]$. This is equivalent to requiring that $(n-1)[1_A] = 0$. \hfill \Box

As a corollary of Theorem 1.1 we have that $[X, \text{Aut}(O_\infty)]$ reduces to a point. The homotopy groups of the endomorphisms of the stable Cuntz-Krieger algebras were computed in [7]. Let $v_1, \ldots, v_n$ be the canonical generators of $O_n$, $2 \leq n < \infty$.

**Theorem 7.4.** For any compact metrizable space $X$ there is a bijection $[X, \text{Aut}(O_n)] \to K_1(C(X) \otimes O_n)$. The $k^{th}$-homotopy group $\pi_k(\text{Aut}(O_n))$ is isomorphic to $\mathbb{Z}/(n-1)$ if $k$ is odd and it vanishes if $k$ is even. In particular $\pi_1(\text{Aut}(O_n))$ is generated by the class of the canonical action of $\mathbb{T}$ on $O_n$, $\lambda_z(v_i) = zv_i$.

Proof. Since $O_n$ satisfies the UCT, we deduce that $\text{End}(O_n)^* = \text{End}(O_n)$. An immediate application of Proposition 6.1 shows that the natural map $\text{Aut}(O_n) \to \text{End}(O_n)$ induces an isomorphism of groups $[X, \text{Aut}(O_n)] \cong [X, \text{End}(O_n)]$. Let $i : O_n \to C(X) \otimes O_n$ be defined by $i(v_i) = 1_{C(X)} \otimes v_i$, $i = 1, \ldots, n$. The map $\psi \mapsto u(\psi) = \psi(v_1)u(v_1)^* + \cdots + \psi(v_n)u(v_n)^*$ is known to be a homeomorphism from $\text{Hom}(O_n, C(X) \otimes O_n)$ to the unitary group of $C(X) \otimes O_n$. Its inverse maps a unitary $u$ to the \*homomorphism $\psi$ uniquely defined by $\psi(v_i) = u(v_i)$, $i = 1, \ldots, n$. Therefore

$$[X, \text{Aut}(O_n)] \cong [X, \text{End}(O_n)] \cong \pi_0(U(C(X) \otimes O_n)) \cong K_1(C(X) \otimes O_n).$$

The last isomorphism holds since $\pi_0(U(B)) \cong K_1(B)$ if $B \cong B \otimes O_\infty$, by [28, Lemma 2.1.7]. One verifies easily that if $\varphi \in \text{Hom}(O_n, C(X) \otimes O_n)$, then $u(\varphi) = \psi(u(\varphi))u(\psi)$. Therefore the bijection $\chi : [X, \text{End}(D)] \to K_1(C(X) \otimes O_n)$ is an isomorphism of groups whenever $K_1(\psi) = \text{id}$ for all $\psi \in \text{Hom}(O_n, C(X) \otimes O_n)$. Using the $C(X)$-linearity of $\psi$ one observes that this holds if the $n-1$ torsion of $K_0(C(X))$ reduces to $[0]$, since in that case the map $K_1(C(X)) \to K_1(C(X) \otimes O_n)$ is surjective by the Künneth formula. \hfill \Box
Corollary 7.5. Let $X$ be a finite dimensional compact metrizable space. The isomorphism classes of unital separable $C(SX)$-algebras with all fibers isomorphic to $\mathcal{O}_n$ are parameterized by $K_1(C(X)\otimes \mathcal{O}_n)$.

Proof. This follows from Theorems 1.1 and 7.4, since the locally trivial principal $H$-bundles over $SX = X \times [0,1]/X \times \{0,1\}$ are parameterized by the homotopy classes $[X,H]$ if $H$ is a path connected group [17, Cor. 8.4]. Here we take $H = \text{Aut}(\mathcal{O}_n)$.

Examples of nontrivial unital $C(X)$-algebras with fiber $\mathcal{O}_n$ over a $2m$-sphere arising from vector bundles were exhibited in [36], see also [35].

We need some preparation for the proof of Theorem 1.5. Let $G$ be a group, let $g \in G$ and set $\text{End}(G,g) = \{\alpha \in \text{End}(G) : \alpha(g) = g\}$. The pair $(G,g)$ is called weakly rigid if $\text{End}(G,g) \subset \text{Aut}(G)$ and rigid if $\text{End}(G,g) = \{\text{id}_G\}$.

Theorem 7.6. If $G$ is a finitely generated abelian group, then $(G,g)$ is weakly rigid if and only if $(G,g)$ is isomorphic to one of the pointed groups from the list $\mathcal{G}$ of Theorem 1.5.

Proof. First we make a number of remarks.

(1) $(G,g)$ is weakly rigid if and only if $(G,\alpha(g))$ is weakly rigid for some (or any) $\alpha \in \text{Aut}(G)$. Indeed if $\beta \in \text{End}(G,g)$ then $\alpha \beta \alpha^{-1} \in \text{End}(G,\alpha(g))$.

(2) By considering the zero endomorphism of $G$ we see that if $(G,g)$ is weakly rigid and $G \neq 0$ then $g \neq 0$.

(3) If $(G \oplus H, g \oplus h)$ is weakly rigid, then so are $(G,g)$ and $(H,h)$.

(4) Let us observe that $(\mathbb{Z}^2,g)$ is not weakly rigid for any $g$. Indeed, if $g = (a,b) \neq 0$, then the matrix $\begin{pmatrix} 1 + b^2 & -ab \\ -ab & 1 + a^2 \end{pmatrix}$ defines an endomorphism $\alpha$ of $\mathbb{Z}^2$ such that $\alpha(g) = g$, but $\alpha$ is not invertible since $\det(\alpha) = 1 + a^2 + b^2 > 1$.

(5) Let $p$ be a prime and let $1 \leq e_1 \leq e_2$, $0 \leq s_1 < e_1$, $0 \leq s_2 < e_2$ be integers. If $(G,g) = (\mathbb{Z}/p^{s_1} \oplus \mathbb{Z}/p^{s_2}, p^{s_1} \oplus p^{s_2})$ is weakly rigid then $0 < s_2 - s_1 < e_2 - e_1$. Indeed if $s_1 \geq s_2$ then the matrix $\begin{pmatrix} 0 & p^{s_1-s_2} \\ p^{s_1-s_2} & 1 \end{pmatrix}$ induces a noninjective endomorphism of $(G,g)$. Also if $s_1 < s_2$ and $s_2 - s_1 \geq e_2 - e_1$ then $p^{s_1}b = 0$ in $\mathbb{Z}/p^{s_2}$, where $b = p^{e_2-s_1}$ and so the matrix $\begin{pmatrix} 1 & 0 \\ b & 0 \end{pmatrix}$ induces a well-defined noninjective endomorphism of $(G,g)$.

(6) Let $p$ be a prime and let $1 \leq k$, $0 \leq s < e$ be integers. Suppose that $(\mathbb{Z} \oplus \mathbb{Z}/p^e, k \oplus p^s)$ is weakly rigid. Then $k$ is divisible by $p^{s+1}$. Indeed, seeking a contradiction suppose that $k$ can be written as $k = p^tc$ where $0 \leq t \leq s$ and $c$ are integers such that $c$ is not divisible by $p$. Let $d$ be an integer such that $dc - 1$ is divisible by $p^s$. Then the matrix $\begin{pmatrix} 1 & 0 \\ dp^{s-t} & 0 \end{pmatrix}$ induces a noninjective endomorphism of $(\mathbb{Z} \oplus \mathbb{Z}/p^e, k \oplus p^s)$.

Suppose now that $(G,g)$ is weakly rigid. We shall show that $(G,g)$ is isomorphic to one of the pointed groups from the list $\mathcal{G}$. Since $G$ is abelian and finitely generated it decomposes as a direct sum of its primary components

$$ G \cong \mathbb{Z}^r \oplus G(p_1) \oplus \cdots \oplus G(p_m) $$
where $p_i$ are distinct prime numbers. Each primary component $G(p_i)$ is of the form

\[(21) \quad G(p_i) = \mathbb{Z}/p_i^{e_i} \oplus \cdots \oplus \mathbb{Z}/p_i^{e_i n(i)}\]

where $1 \leq e_1 \leq \cdots \leq e_i n(i)$ are positive integers. Corresponding to the decomposition (20) we write the base point $g = g_0 \oplus g_1 \oplus \cdots \oplus g_m$ with $g_i \in \mathbb{Z}^r$ and $g_i \in G(p_i)$ for $i \geq 1$. If $g_{ij}$ is the component of $g_i$ in $\mathbb{Z}/p_i^{e_i}$, then it follows from (1), (2) and (3) that we may assume that $g_{ij} = p_i^{s_{ij}}$ for some integer $0 \leq s_{ij} < e_{ij}$. Using (3) and (4) we deduce that $r = 1$ in (20) and that $g_0 = k \neq 0$ by (2). We may assume that $k \geq 1$ by (1). Then using (3) and (5) we deduce that for each $1 \leq i \leq m, 0 < s_{i,j+1} - s_{i,j} < e_{i,j+1} - e_{i,j} \leq 1 \leq j < n(i)$. Finally, from (3) and (6) we see that $k$ is divisible by the product $p_1^{s_{1,n(1)}} \cdots p_m^{s_{m,n(m)}}$. Therefore $(G, g)$ is isomorphic to one of the pointed groups on the list $\mathcal{G}$.

Conversely, we shall prove that if $(G, g)$ belongs to the list $\mathcal{G}$ then $(G, g)$ is weakly rigid. This is obvious if $G$ is torsion free i.e. for $(\{0\}, 0)$ and $(\mathbb{Z}, k)$ with $k \geq 1$.

Let us consider the case when $G$ is a torsion group. Since

$$\text{End}(G(p_1) \oplus \cdots \oplus G(p_m), g_1 \oplus \cdots \oplus g_m) \cong \bigoplus_{i=1}^{m} \text{End}(G(p_i), g_i)$$

it suffices to assume that $G$ is a $p$-group,

\[(G, g) = (\mathbb{Z}/p^{e_1} \oplus \cdots \oplus \mathbb{Z}/p^{e_n}, p^{s_1} \oplus \cdots \oplus p^{s_n})\]

with $0 \leq s_i < e_i$ for $i = 1, \ldots, n$ and $0 < s_{i+1} - s_i < e_{i+1} - e_i$ for $1 \leq i < n$. For each $0 \leq i, j \leq n$ set $e_{ij} = \max\{e_i - e_j, 0\}$. It follows immediately that $s_i < e_{ij} + s_j$ for all $i \neq j$. Let $\alpha \in \text{End}(G, g)$.

It is well-known that $\alpha$ is induced by a square matrix $A = [a_{ij}] \in M_n(\mathbb{Z})$ with the property that each entry $a_{ij}$ is divisible by $p^{e_i}$ and so $a_{ij} = p^{s_{ij}}b_{ij}$ for some $b_{ij} \in \mathbb{Z}$, see [16]. Since $\alpha(g) = g$, we have $\sum_{j=1}^{n} b_{ij}p^{s_{ij} + s_j} = p^{s_i}$ in $\mathbb{Z}/p^{e_i}$ for all $0 \leq i \leq n$. Since $e_{ij} + s_j > s_i$ for $i \neq j$ and $e_i > s_i$ we see that $b_{ij} - 1$ must be divisible by $p$ for all $1 \leq i \leq n$. Since det$(A)$ is congruent to $b_{11} \cdots b_{nn}$ modulo $p$ it follows that det$(A)$ is not divisible by $p$ and so $\alpha \in \text{Aut}(G)$ by [16].

Finally consider the case when $(G, g) = (\mathbb{Z} \oplus G(p_1) \oplus \cdots \oplus G(p_m), k \oplus g_1 \oplus \cdots \oplus g_m)$. If $\gamma \in \text{End}(G, g)$ then there exist $\alpha_i \in \text{End}(G(p_i), g_i)$ and $d_i \in G(p_i)$, $1 \leq i \leq n$, such that $\gamma(x_0 \oplus x_1 \oplus \cdots \oplus x_m) = x_0 \oplus (\alpha_1(x_1) + x_0d_1) \oplus \cdots \oplus (\alpha_m(x_1) + x_0d_m)$. Note that if each $\alpha_i$ is an automorphism then so is $\gamma$. Indeed, its inverse is $\gamma^{-1}(x_0 \oplus x_1 \oplus \cdots \oplus x_m) = x_0 \oplus (\alpha_1^{-1}(x_1) + x_0c_1) \oplus \cdots \oplus (\alpha_m(x_1)^{-1} + x_0c_m)$, where $c_i = -\alpha_{-1}^{-1}(d_i)$. Therefore it suffices to consider the case $m = 1$, i.e.

\[(G, g) = (\mathbb{Z} \oplus \mathbb{Z}/p^{e_1} \oplus \cdots \oplus \mathbb{Z}/p^{e_n}, k \oplus p^{s_1} \oplus \cdots \oplus p^{s_n}),\]

and $(G, g)$ is on the list $\mathcal{G}$ (e). In particular $k = p^{s_n+1} \ell$ for some $\ell \in \mathbb{Z}$. Let $\gamma \in \text{End}(G, g)$. Then there exists $\alpha \in \text{End}(G(p))$ and $d \in G(p)$ such that $\gamma(x_0 \oplus x) = x_0 \oplus (\alpha(x) + x_0d)$. Just as above, $\alpha$ is induced by a square matrix $A \in M_n(\mathbb{Z})$ of the form $A = [b_{ij}p^{e_i}] \in M_n(\mathbb{Z})$ with $b_{ij} \in \mathbb{Z}$, $e_{ij} = \max\{e_i - e_j, 0\}$ and $d \in \mathbb{Z}^n$. Since $\gamma(g) = g$ we have that $p^{s_n+1}\ell d + \sum_{j=1}^{n} b_{ij}p^{e_j} = p^{s_i}$ in $\mathbb{Z}/p^{e_i}$ for all $0 \leq i \leq n$, where the $d_i$ are the components of $d$. By reasoning as in the case when $G$ was a torsion group considered above, since $s_n + 1 > s_i$ for all $1 \leq i \leq n$, $e_{ij} + s_i > s_i$ for all $i \neq j$ and $e_i > s_i$, it follows again that each $b_{ij} - 1$ is divisible by $p$ and that the endomorphism $\alpha$ of $G(p)$ induced by the matrix $A$ is an automorphism. We conclude that $\gamma$ is an automorphism. □

Proof of Theorem 1.5
Proof. (ii) and (iii) Let \( D \) be a unital Kirchberg algebra such that \( D \) satisfies the UCT and \( K_* (D) \) is finitely generated. Then \( D \) is KK-semiprojective by Proposition 3.14 and \( KK(D,D)^{-1} = \{ \alpha \in KK(D,D) : K_*(\alpha) \text{ is bijective} \} \). By Theorem 3.1, all unital endomorphisms of \( D \) are KK-equivalences if and only if \( (K_0(D) \oplus K_1(D), [1_D] \oplus 0) \) is weakly rigid. Equivalently, \( K_1(D) = 0 \) and \( (K_0(D), [1_D]) \) is weakly rigid. By Theorem 7.6 \( (K_0(D), [1_D]) \) is weakly rigid if and only if it isomorphic to one pointed groups from the list \( G \) of Theorem 1.5. We conclude the proof of (ii) and (iii) by applying Theorem 1.4.

(i) By Theorem 1.1 both \( O_2 \) and \( O_\infty \) have the automatic triviality property. Conversely, let \( D \) be a unital Kirchberg algebra such that \( D \) satisfies the UCT and \( K_* (D) \) is finitely generated and suppose that \( D \) has the automatic triviality property. We shall prove that \( D \) is isomorphic to either \( O_2 \) or \( O_\infty \).

Let \( Y \) be a finite connected CW-complex and let \( \iota : D \to C(Y) \otimes D \) be the map \( \iota (d) = 1 \otimes d \). Let \([D,C(Y) \otimes D]\) denote the homotopy classes of unital \(*\)-homomorphisms from \( D \) to \( C(Y) \otimes D \).

By Theorem 3.1 the image of the map \( \Delta : [D,C(Y) \otimes D] \to KK(D,C(Y) \otimes D) \) defined by \( [\varphi] \mapsto KK(\varphi) - KK(\iota) \) coincides with the kernel of the restriction morphism \( \rho : KK(D,C(Y) \otimes D) \to KK(C1_D,C(Y) \otimes D) \).

We claim that \( \ker \rho \) must vanish for all \( Y \). Let \( h \in \ker \rho \). Then there is a unital \(*\)-homomorphism \( \varphi : D \to C(Y) \otimes D \) such that \( \Delta (\varphi) = h \). By Theorem 1.4, each unital endomorphism of \( D \) induces a KK-equivalence. Therefore, by Proposition 6.1 there is a \(*\)-homomorphism \( \Phi : D \to C(Y) \otimes D \) such that \( \Phi_y \in \text{Aut}(D) \) for all \( y \in Y \) and \( KK(\Phi) = KK(\varphi) \). Therefore \( \Delta (\Phi) = KK(\Phi) - KK(\iota) = h \). By hypothesis, the \( \text{Aut}(D)\)-principal bundle constructed over the suspension of \( Y \) with characteristic map \( \varphi \to \Phi_y \) is trivial. It follows then from [17, Thm. 8.2 p85] that this map is homotopic to the to the constant map \( Y \to \text{Aut}(D) \) which shrinks \( Y \) to \( \text{id}_D \). This implies that \( \Phi \) is homotopic to \( \iota \) and hence \( h = 0 \).

Let us now observe that \( \ker \rho \) contains subgroups isomorphic to \( \text{Hom}(K_1(D),K_1(D)) \) and \( \text{Ext}(K_0(D),K_0(D)) \) if \( Y = T \), since \( D \) satisfies the UCT. It follows that both these groups must vanish and so \( K_1(D) = 0 \) and \( K_0(D) \) is torsion free. On the other hand, \( (K_0(D), [1_D]) \) is weakly rigid by the first part of the proof. Since \( K_0(D) \) is torsion free we deduce from Theorem 7.6 that either \( K_0(D) = 0 \) in which case \( D \cong O_2 \) or that \( (K_0(D), [1_D]) \cong (Z,k) \), \( k \geq 1 \), in which case \( D \cong M_k (O_\infty) \) by the classification theorem of Kirchberg and Phillips.

To conclude the proof, it suffices to show that \( \ker \rho \neq 0 \) if \( D = M_k (O_\infty) \), \( k \geq 2 \) and \( Y \) is the two-dimensional space obtained by attaching a disk to a circle by a degree-\( k \) map. Since \( K_0(C(Y) \otimes O_\infty) \cong Z \oplus Z/k \) we can identify the map \( \rho \) with the map \( Z \oplus Z/k \to Z \oplus Z/k \) and so \( \ker \rho \cong Z/k \).

\[ \square \]

Added in proof. Some of the results from this paper are further developed in [9]. Theorem 1.2 was shown to hold for all stable Kirchberg algebras \( D \). The assumption that \( X \) is finite dimensional is essential Theorem 1.1. Theorem 1.5 (ii) extends as follows: \( O_2, O_\infty \) and \( B \otimes O_\infty \) where \( B \) is a unital UHF algebras of infinite type are the only unital Kirchberg algebras which satisfy the UCT and have the automatic triviality property.

References


