THE JIANG-SU ALGEBRA DOES NOT ALWAYS EMBED

MARIUS DADARLAT, ILAN HIRSHBERG, ANDREW S. TOMS, AND WILHELM WINTER

Abstract. We exhibit a unital simple nuclear non-type-I $C^*$-algebra into which the Jiang-Su algebra does not embed unitally. This answers a question of M. Rørdam.

The Jiang-Su algebra, denoted by $Z$ ([3]), occupies a central position in the structure theory of separable amenable $C^*$-algebras. The property of absorbing the Jiang-Su algebra tensorially is a necessary, and, in considerable generality, sufficient condition for the confirmation of G. A. Elliott's K-theoretic rigidity conjecture for simple separable amenable $C^*$-algebras ([5], [7]). The uniqueness question for this algebra is therefore of great interest. M. Rørdam observed that if $C$ is a class of unital separable $C^*$-algebras, and $A \in C$ has the properties that (i) for every $B \in C$ there is a unital $*$-homomorphism $\gamma : A \to B$ and (ii) every unital $*$-endomorphism of $A$ is approximately inner, then $A$ is the only such algebra, up to isomorphism. (This follows from an application of Elliott's Intertwining Argument.) Every unital $*$-endomorphism of $Z$ is approximately inner ([3]), and there are no obvious obstructions to the existence of a unital $*$-homomorphism $\gamma : Z \to A$ for any unital separable $C^*$-algebra $A$ without finite-dimensional quotients. Indeed, such a $\gamma$ always exists when $A$ has real rank zero, and examples show that the existence of $\gamma$ is strictly weaker than tensorial absorption of $Z$—see [1] and [6], respectively. All of this begs the question, first posed by Rørdam: "Does every unital $C^*$-algebra without finite-dimensional quotients admit a unital embedding of $Z$?", see [1]. We prove that the answer is negative, even when the target algebra is simple and nuclear.

Theorem. There is a unital simple nuclear infinite dimensional $C^*$-algebra (in fact, an AH algebra) into which the Jiang-Su algebra does not embed unitally.

Date: January 4, 2008.

2000 Mathematics Subject Classification. Primary 46L35, Secondary 46L80.

Key words and phrases. Jiang-Su algebra, embeddability.

The authors were partially supported by the Fields Institute; M.D. was partially supported by NSF grant #DMS-0500693; I.H. was partially supported by the Israel Science Foundation (grant No. 1471/07); A.T. was partially supported by NSERC.
In the remainder of the paper we give some background discussion and prove
the theorem. For a pair of relatively prime integers \( p, q > 1 \), we set

\[
Z_{p,q} = \{ f \in C([0, 1]; M_p \otimes M_q) \mid f(0) \in M_p \otimes 1, f(1) \in 1 \otimes M_q \}.
\]

Each \( Z_{p,q} \) is contained unitally in \( Z \). If a unital \( \ast \)-algebra \( A \) admits no unital
\( \ast \)-homomorphism \( \gamma : Z_{p,q} \to A \), then there is no unital embedding of \( Z \) into \( A \).

Let \( A, B \) be unital \( \ast \)-algebras, and let \( e, f \in A \) be projections satisfying \((n + 1)[e] \leq n[f] \) in the Murray-Von Neumann semigroup \( V(A) \) for some \( n \in \mathbb{N} \). It
is implicitly shown in the proof of [4, Lemma 4.3] that if \( \gamma : Z_{n,n+1} \to B \) is a unital
\( \ast \)-homomorphism, then \( [e \otimes 1_B] \leq [f \otimes 1_B] \) in \( V(A \otimes B) \); tensor products
are minimal. Example 4.8 of [2] exhibits a sequence \( (B_j)_{j \in \mathbb{N}} \) of unital separable
\( \ast \)-algebras with the following property: there are projections \( e, f \in B_1 \otimes B_2 \) such
that \( 4[e] \leq 3[f] \), but \( [e \otimes 1_{\otimes_{j=3}^n B_j}] \not\leq [f \otimes 1_{\otimes_{j=3}^n B_j}] \) for any \( n \geq 3 \). Using Rørdam’s
result, one concludes that there is no unital \( \ast \)-homomorphism \( \gamma : Z_{3,4} \to \otimes_{j=3}^n B_j \)
for any \( j \geq 3 \). (In fact, there is nothing special about \( Z_{3,4} \). A similar construction
can be carried out for a wide variety of \( Z_{p,q} \).)

To simplify notation, we renumber the \( B_j \)’s so that \( Z_{3,4} \) does not embed into
\( \otimes_{j=1}^n B_j \) for any \( n \in \mathbb{N} \). For each \( i \in \mathbb{N} \), set \( D_i = \otimes_{j=1}^i B_j \). We will perturb the canonical embeddings

\[
\psi_i := \text{id} \otimes 1_{B_{i+1}} : D_i \to D_i \otimes B_{i+1} = D_{i+1}
\]

to maps \( \phi_i \) with the property that \( (D_i, \phi_i)_{i \in \mathbb{N}} \) has simple limit \( D \). Any such limit,
simple or not, fails to admit a unital \( \ast \)-homomorphism \( \gamma : Z_{3,4} \to D \), and so also
fails to admit a unital embedding of \( Z \). Indeed, suppose that such a \( \gamma \) did
exist. Then, by the semiprojectivity of \( Z_{3,4} \) ([3]), there would exist a unital \( \ast \)-homomorphism \( \tilde{\gamma} : Z_{3,4} \to D_i \) for some \( i \), contradicting our choice of \( D_i \). We
remark that, in particular, \( \otimes_{j=1}^\infty B_j \) admits no unital embedding of \( Z \). This algebra
is a continuous field of \( \ast \)-algebras whose fibres are \( Z \)-absorbing – in fact, its
fibres are all isomorphic to the CAR algebra (see [2, Example 4.8]).

The \( B_j \)’s have the form \((e_j \oplus f_j)(C(X_j) \otimes K)(e_j \oplus f_j)\), where \( e_j \) and \( f_j \) are rank one
projections and \( X_j = (S^2)^{\times m(j)} \). Let \( \alpha : X_j \to X_j \) be a homeomorphism homotopic
to the identity map, and view \( B_j \) as a corner of \( C(X_j) \otimes M_n \) for some sufficiently
large \( n \in \mathbb{N} \). The map \( \alpha \) induces an automorphism \( \alpha^* \) of \( C(X_j) \otimes M_n \), \( \alpha^*(f) = f \circ \alpha \). In general, \( \alpha^* \) will not carry \( B_j \) into \( B_j \), but this can be corrected. Since \( \alpha \) is
homotopic to the identity, the projection \( e_j \oplus f_j \) is homotopic, and hence unitarily
equivalent, to its image under \( \alpha^* \). If \( u \) is a unitary implementing this equivalence,
then \( \overline{\alpha} := (\text{Ad}(u) \circ \alpha^*)|_{B_j} \) is an automorphism of \( B_j \). For our purposes, the salient
property of $\pi$ is this: if $f \in B_j$ and $f(x) \neq 0$ for some $x \in X_j$, then $\bar{\pi}(f)(\alpha^{-1}(x)) \neq 0$.

It remains to construct the $\phi_i$, and prove the simplicity of the resulting inductive limit algebra $D$. Let us set $Y_i := \prod_{j=1}^{i} X_j$ where $i \in \mathbb{N}$ or $i = \infty$. We endow $Y_i$ with the metric $d(x, y) = \sum_{j=1}^{i} 2^{-j} d_j(x^j, y^j)$ where $d_j$ is the canonical metric on $X_j$, normalized so that $X_j$ has diameter equal to one.

Choose a dense sequence $(z_k)_{k \in \mathbb{N}}$ in $Y_\infty$. Fix $z_0 \in Y_\infty$ and for each $k \in \mathbb{N}$ let $\beta_k : Y_\infty \to Y_\infty$ be a cartesian product of isometries of $X_j$s which are homotopic to the identity and such that $\beta_k(z_0) = z_k$. Let $(\alpha_k)_{k \in \mathbb{N}}$ be an enumeration of the set $\{\beta_m\beta^{-1}_n : n, m \in \mathbb{N}\}$. It is easy to see that for any point $x \in Y_\infty$ and any $i \in \mathbb{N}$, the sequence $(\alpha_k(x))_{k \geq i}$ is dense in $Y_\infty$. Note that each $\alpha_k$ is also a cartesian product of isometries $\alpha^k_j$ of $X_j$ homotopic to $\text{id}_{X_j}$. Let us set $\alpha_{k,[i]} = \prod_{j=1}^{i} \alpha^k_j$. Let $\pi_i : Y_\infty \to Y_i$ be the co-ordinate projection. Then $\pi_i \alpha_k(x) = \alpha_{k,[i]}(\pi_i(x))$ for $x \in Y_\infty$. Therefore for any point $y \in Y_i$, the sequence $(\alpha_{k,[i]}(y))_{k \geq i}$ is dense in $Y_i$. So is the sequence $((\alpha_{k,[i]}(y))_{k \geq i}$ since each $\alpha_{k,[i]}$ is an isometry. By the compactness of $Y_i$ it follows that for any nonempty open set $U$ of $Y_i$, there is $j \geq i$ such that $Y_i = \bigcup_{k=i}^{j} (\alpha_{k,[i]})^{-1}(U)$.

For each $i \leq k \in \mathbb{N}$, let $\alpha_{k,[i]} : D_j \to D_i$ be the automorphism induced, in the manner described above, by the homeomorphism $\alpha_{k,[i]} : Y_i \to Y_i$.

Observe that the canonical embedding $\psi_i : D_i \to D_{i+1}$ is the direct sum of two non-unital embeddings:

$$\psi_i^{(1)} \overset{\text{def}}{=} \text{id} \otimes e_{i+1} : D_i \to D_i \otimes e_{i+1} \subseteq D_{i+1},$$

and

$$\psi_i^{(2)} \overset{\text{def}}{=} \text{id} \otimes f_{i+1} : D_i \to D_i \otimes f_{i+1} \subseteq D_{i+1}.\,$$

Set $\phi_i^{(1)} = \psi_i^{(1)}$, and

$$\phi_i^{(2)} \overset{\text{def}}{=} \alpha_{i,[i]} \otimes f_{i+1} : D_i \to D_i \otimes f_{i+1} \subseteq D_{i+1}.\,$$

Define $\phi_i : D_i \to D_{i+1}$ to be $\phi_i^{(1)} \oplus \phi_i^{(2)}$.

Let us now verify that $D = \lim_{i \to \infty}(D_i, \phi_i)$ is simple. It will suffice to prove that for any nonzero $a \in D_i$ there is some $j \geq i$ such that $\phi_{i,j+1}(a) := \phi_j \circ \cdots \circ \phi_i(a)$ is nonzero over every point in the spectrum of $D_{j+1}$.

For each $v = (v_i, \ldots, v_j) \in \{1, 2\}^{j-i+1}$, set

$$\phi^{v}_{i,j+1} = \phi_{j}^{v_j} \circ \phi_{j-1}^{v_{j-1}} \circ \cdots \circ \phi_{i}^{v_i},$$

and note that $\phi_{i,j+1} = \bigoplus_{v \in \{1,2\}^{j-i+1}} \phi^{v}_{i,j+1}$. For $k \in \{i, \ldots, j\}$, let $v_k \in \{1, 2\}^{j-i+1}$ be the vector which is equal to 1 in each co-ordinate except the $k^{th}$ one. We have (with
the exception of the cases $k = i, j$ when the formula reads slightly differently)
\[
\phi_{i,j+1}^*(a) = \left[ \alpha_{k,[k]}(a \otimes e_{i+1} \otimes \cdots \otimes e_k) \right] \otimes f_{k+1} \otimes e_{k+2} \otimes \cdots \otimes e_{j+1}
\]
\[
= \left[ \alpha_{k,[i]}(a) \otimes \alpha_{k}^{i+1}(e_{i+1}) \otimes \cdots \otimes \alpha_k^{j}(e_k) \right] \otimes f_{k+1} \otimes e_{k+2} \otimes \cdots \otimes e_{j+1}.
\]

Since $a$ is nonzero on some nonempty open set $U$, the formula above shows that $\phi_{i,j+1}^*(a)$ is nonzero on $W_{i,j+1}^k := (\alpha_{k,[i]})^{-1}(U) \times X_{i+1} \times \cdots \times X_{j+1}$, for any $j \geq i$. As noticed earlier, there is $j \geq i$ such that $Y_i = \bigcup_{k=i}^j (\alpha_{k,[i]})^{-1}(U)$. Therefore $\bigoplus_{k=i}^j \phi_{i,j+1}^*(f)$ is nonzero on $\bigcup_{k=i}^j W_{i,j+1}^k = Y_{j+1} = \hat{D}_{j+1}$, as required.

REFERENCES


DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, 150 N. UNIVERSITY ST., WEST LAFAYETTE, IN, 47907-2067, USA
E-mail address: mdd@math.purdue.edu

DEPARTMENT OF MATHEMATICS, BEN GURION UNIVERSITY OF THE NEGEV, P.O.B. 653, BE’ER SHEVA 84105, ISRAEL
E-mail address: ilan@math.bgu.ac.il

DEPARTMENT OF MATHEMATICS AND STATISTICS, YORK UNIVERSITY, 4700 Keele St., TORONTO, ONTARIO, CANADA, M3J 1P3
E-mail address: atoms@mathstat.yorku.ca

SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF NOTTINGHAM, UNIVERSITY PARK, NOTTINGHAM, NG7 2RD, UNITED KINGDOM
E-mail address: wilhelm.winter@nottingham.ac.uk