EMBEDDINGS OF NUCLEARLY EMBEDDABLE C*-ALGEBRAS

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Abstract. Let B be a unital simple C*-algebra and let $\mathcal{U} = \bigotimes_{n=1}^{\infty} M_n(\mathbb{C})$ be the universal UHF algebra. We give sufficient conditions for a nuclearly embeddable C*-subalgebra of $\prod_{n=1}^{\infty} B$ to embed into $B \otimes \mathcal{U}$. In particular we prove that a nuclearly embeddable residually finite-dimensional C*-algebra A is embeddable in $\mathcal{U}$, provided that either the Hausdorff quotient of the rational K-homology of A is finitely generated, or A satisfies the Universal Coefficient Theorem (UCT) for the Kasparov groups. This yields a new proof of Kirchberg's characterization of separable nuclearly embeddable C*-algebras as subquotients of $\mathcal{U}$. It also implies that the C*-algebra of a second countable locally compact amenable maximally almost periodic group embeds in $\mathcal{U}$. More generally, if a discrete countable amenable group Γ embeds in a product $\prod_{n=1}^{\infty} U(B_n)$ of unitary groups of simple unital quasidiagonal C*-algebras B_n and $B = (\bigotimes_{n=1}^{\infty} B_n) \otimes \mathcal{U}$ has bounded exponential length, then $C^*(\Gamma)$ embeds in $\bigotimes_{n=1}^{\infty} B$.

1. Introduction

In recent years there have been spectacular advances in the structure theory of C*-algebras. A C*-algebra A is called *nuclearly embeddable* if there is a C*-algebra C and a nuclear $*$-monomorphism $A \to C$. Equivalently, any completely positive map $\sigma : A \to \mathcal{L}(\mathcal{H})$ is nuclear [Vo3]. S. Wassermann [W1] has shown that any nuclearly embeddable C*-algebra is exact. By an important theorem of Kirchberg, the converse is also true: any exact C*-algebra is nuclearly embeddable [Ki2]. Moreover, Kirchberg has shown that a separable C*-algebra is nuclearly embeddable if and only if it embeds in the Cuntz algebra $\mathcal{O}_2$ [Ki4]. The property of $\mathcal{O}_2$ of being infinite is essential here not only to accommodate embeddings of infinite exact algebras but also for deeper structural reasons. Indeed, Kirchberg has pointed out that there are stably finite exact C*-algebras (such as the reduced C*-algebra of the free group on two generators) which do not embed in any stably finite nuclear C*-algebra [Ki5].

A major open problem is to characterize the C*-subalgebras of AF algebras. In view of the very interesting results on AF embeddings of [PV], [Pi], [Vo2], [Sp], [Br1, Br2] and [L1], it is natural to conjecture that a separable C*-algebra embeds in an AF algebra if and only if it is quasidiagonal and nuclearly embeddable [BK], [Br3].

In this paper we give sufficient conditions for a separable nuclearly embeddable C*-subalgebra of $\prod_{n=1}^{\infty} B_n$ to embed in $\bigotimes_{n=1}^{\infty} B_n \otimes \mathcal{U}$, where $\mathcal{U} = \bigotimes_{n=1}^{\infty} M_n(\mathbb{C})$ is the universal UHF algebra (Theorem 1.1). In view of Kirchberg's results which give embeddings into infinite C*-algebras, the most interesting cases are those when B is stably finite, quasidiagonal or AF. As corollaries, we prove that any nuclear embeddable residually finite-dimensional C*-algebra which satisfies the UCT

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embeds in $\mathcal{U}$ (Corollary 1.2) and that if A is a separable nuclearly embeddable quasidiagonal C*-algebra satisfying the UCT, then $A + \mathcal{K}(\mathcal{H})$, the essential trivial extension of A by the compacts, is the closure of an increasing sequence of C*-algebras embeddable in $\mathcal{U}$ (Corollary 1.4). We do not rely on the equivalence between nuclear embeddability, subnuclearity and exactness. In fact, Kirchberg's fundamental characterization of nuclearly embeddable separable C*-algebras as subquotients of UHF algebras [Ki2] is an immediate consequence of Corollary 1.2 below (see also [D4] where we distill our arguments to a completely elementary proof which does not use K-theory). In a different direction, we obtain embedding results for C*-algebras of certain amenable groups, see Corollary 1.5, Theorem 1.6 and Corollary 1.7.

Our methods rely on KK-theory [Kas1], [Sk], [DE2] and approximation results for nuclearly embeddable C*-algebras in the spirit of [D3]. The main results are Theorems 3.1–3.2. Some of their consequences are discussed in more detail in the remaining of the introduction.

We refer the reader to [Sk] for a discussion of $\mathrm{KK}_{nuc}(A, B)$, the nuclear version of the Kasparov groups. The group $\mathrm{KK}_{nuc}(A, B)$ admits a natural topology. Its Hausdorff quotient is denoted by $\widehat{\mathrm{KK}}_{nuc}(A, B)$, see Definition 2.3 and Remark 2.5. Throughout the paper we consider only minimal tensor products [Ta]. We say that a sequence (x_n) is an *infinite-multiplicity* sequence if each element x_n occurs infinitely many times.

Theorem 1.1. *Let (B_n) be an infinite-multiplicity sequence of unital simple C*-algebras and let B denote either the infinite tensor product $\bigotimes_{n=1}^{\infty} B_n$ or $B = B_1$ if B_n are mutually isomorphic. Let A be a separable unital C*-algebra which admits a unital nuclear embedding $A \hookrightarrow \prod_{n=1}^{\infty} B_n$. Suppose that A is quasidiagonal, satisfies the UCT* [RS] *and $B \otimes \mathcal{U}$ has bounded exponential length. Then there is a unital nuclear embedding $A \hookrightarrow B \otimes \mathcal{U}$.*

The theorem remains true if we don't require A to be quasidiagonal and satisfy the UCT, but instead we assume that there is a sequence of C*-subalgebras (A_i) of A (not necessarily nested) whose union is dense in A and such that the vector spaces $\widehat{\mathrm{KK}}_{nuc}(A_i, B) \otimes \mathbb{Q}$ are finitely generated. In this case no assumption of the exponential length of $B \otimes \mathcal{U}$ is necessary.

Recall that a separable C*-algebra A is said to satisfy the UCT of [RS] if the sequence

$$0 \to \mathrm{Ext}(K_*(A), K_{*-1}(B)) \to \mathrm{KK}_*(A, B) \to \mathrm{Hom}(K_*(A), K_*(B)) \to 0$$

is exact for any σ-unital C*-algebra B. A unital C*-algebra B is said to have bounded exponential length if there is a constant L such that any two homotopic unitaries in B can be connected by a continuous path of unitaries of length at most L [Ri], [Ph].

A separable C*-algebra A is called *residually finite-dimensional* (abbreviated *RFD*) if it has a separating sequence of finite-dimensional representations. Equivalently, A embeds in a C*-algebra of the form $\prod_{n=1}^{\infty} M_{k(n)}$. The RFD C*-algebras approximate the quasidiagonal C*-algebras in the same way that block-diagonal operators approximate quasidiagonal operators. It was shown in [D2] that a nuclear separable RFD algebra which is homotopically dominated by an AF algebra embeds in an AF algebra. Lin [L2] showed that the nuclear separable RFD algebras satisfying the UCT are AF embeddable. Using Theorem 1.1 we extend those results (in an improved form) to nuclearly embeddable RFD algebras.

Corollary 1.2. *If A is a unital separable nuclearly embeddable RFD C*-algebra satisfying the UCT, then A embeds as a unital C*-subalgebra of $\mathcal{U}$.*

Let $K^0(A) = KK(A, \mathbb{C})$ denote the K-homology of A and $\widehat{K}^0(A) = \widehat{\mathrm{KK}}(A, \mathbb{C})$. The corollary remains true if one replaces the assumption that A satisfies the UCT by the condition that there is a sequence of C*-subalgebras (A_i) of A (not necessarily nested) whose union is dense in A and $\widehat{K}^0(A_i) \otimes \mathbb{Q}$ are finitely generated.

As an application of Corollary 1.2, one obtains a short new proof of Kirchberg's characterization of nuclearly embeddable C*-algebras as subquotients of UHF algebras. Indeed, if C is a separable nuclearly embeddable C*-algebra, it is not hard to show that there is a semisplit short exact sequence $0 \to J \to C_0[0,1) \otimes A \to C \to 0$ with A a nuclearly embeddable RFD algebra [D4]. Since $C_0[0,1) \otimes A$ embeds in $\mathcal{U}$ by Corollary 1.2, this proves that C is a quotient of a subalgebra of $\mathcal{U}$.

Corollary 1.3. [Ki2] *Any separable nuclearly embeddable C*-algebra is a (semisplit) quotient of a C*-subalgebra of* $\mathcal{U}$.

Corollary 1.4. *Let* $A \subset \mathcal{L}(\mathcal{H})$, $A \cap \mathcal{K}(\mathcal{H}) = \{0\}$, *be a separable nuclearly embeddable quasidiagonal C*-algebra satisfying the UCT. Then there is an increasing sequence* (A_i) *of C*-subalgebras of* $A + \mathcal{K}(\mathcal{H})$ *such that* $\overline{\bigcup_i A_i} = A + \mathcal{K}(\mathcal{H})$ *and each* A_i *embeds in* $\mathcal{U}$.

Indeed, one can arrange that the subalgebras A_i are RFD and satisfy the UCT (see Proposition 3.4). The result will then follow from Corollary 1.2. Note that Corollary 1.4 applies to $A = C_0[0,1) \otimes B$ where B is any nuclearly embeddable separable C*-algebra, since $C_0[0,1) \otimes B$ is quasidiagonal by [Vo4].

In a different direction, we give results on embeddings of group C*-algebras. A locally compact group G is called maximally almost periodic (abbreviated MAP) if it has a separating family of finite dimensional unitary representations. Residually finite groups are examples of MAP groups. If G is a second countable amenable locally compact MAP group, then $C^*(G)$ is residually finite dimensional by [BLS] and satisfies the UCT by [Tu]. By Corollary 1.2 we have the following.

Corollary 1.5. *The C*-algebra of a second countable amenable locally compact MAP group* G *is embeddable in* $\mathcal{U}$. *If, moreover,* G *is discrete, then* G *injects in the unitary group of* $\mathcal{U}$.

Corollary 1.5 shows that in general the unitary group of a simple C*-algebra may contain interesting discrete amenable groups. This observation motivates the following result.

Theorem 1.6. *Let* (B_n) *be an infinite-multiplicity sequence of unital simple C*-algebras and let* Γ *be a discrete countable amenable subgroup of* $\prod_{n=1}^{\infty} U(B_n)$. *Suppose that the algebras* B_n *are quasidiagonal and* $(\bigotimes_{n=1}^{\infty} B_n) \otimes \mathcal{U}$ *has bounded exponential length. Then there is a unital embedding* $C^*(\Gamma) \hookrightarrow (\bigotimes_{n=1}^{\infty} B_n) \otimes \mathcal{U}$.

The theorem remains true if we replace the assumption that B_n are quasidiagonal and $(\bigotimes_{n=1}^{\infty} B_n) \otimes \mathcal{U}$ has bounded exponential length by the assumption that there is a sequence of subgroups (Γ_i) of Γ with $\bigcup_{i=1}^{\infty} \Gamma_i = \Gamma$ and such that the vector spaces $K_*(C^*(\Gamma_i)) \otimes \mathbb{Q}$ are finitely generated.

Corollary 1.7. *Let* Γ *be a discrete countable amenable group. The following are equivalent.*

(i) There is a sequence (B_n) *of simple unital separable AF algebras with* $\Gamma \subset \prod_{n=1}^{\infty} U(B_n)$.

(ii) There is a simple unital separable AF algebra B *with* $\Gamma \subset U(B)$.

(iii) There is a simple unital separable AF algebra B *such that* $C^*(\Gamma) \subset B$.

If Γ *satisfies these conditions and* Γ *acts on a compact metrisable space* X *such that the points with finite orbits are dense in* X, *then the crossed-product* $C(X) \times \Gamma$ *embeds in a simple unital AF algebra.*

2. Approximate unitary equivalence and KK-theory

We refer the reader to [Kas1] for a background discussion on Hilbert C*-modules. Let A be a separable C*-algebra and let B be a σ-unital C*-algebra.

Definition 2.1. If $\sigma : A \longrightarrow \mathcal{L}_B(E)$ and $\sigma' : A \longrightarrow \mathcal{L}_B(E')$ are two representations, with E and E' Hilbert B-modules, we say that σ and σ' are *approximately unitarily equivalent* and write $\sigma \simeq \sigma'$, if there exists a sequence of unitaries $u_n \in \mathcal{L}_B(E', E)$ such that

$$\lim_{n\to\infty} \|\sigma(a) - u_n\sigma'(a)u_n^*\| = 0, \quad a \in A \tag{1}$$

$$\sigma(a) - u_n\sigma'(a)u_n^* \in \mathcal{K}_B(E), \quad a \in A \tag{2}$$

We say that the representations σ and σ' are *properly approximately unitarily equivalent* and write $\sigma \approxeq_d \sigma'$, if $E = E'$ and there is a sequence of unitaries $u_n \in \mathbb{C}1_E + \mathcal{K}_B(E)$ satisfying (1) and (2). The equivalence relation $\approxeq_d$ is the discrete version of the equivalence relation $\approxeq$ introduced in [DE2]. The representations $\sigma, \sigma' : A \to \mathcal{L}_B(E)$ are *properly asymptotically unitarily equivalent*, written $\sigma \approxeq \sigma'$, if there is a norm-continuous unitary valued map $u : [0, \infty) \to \mathbb{C}1_E + \mathcal{K}_B(E)$, $t \mapsto u_t$ such that

$$\lim_{t\to\infty} \|\sigma(a) - u_t\sigma'(a)u_t^*\| = 0, \quad a \in A \tag{3}$$

$$\sigma(a) - u_t\sigma'(a)u_t^* \in \mathcal{K}_B(E), \quad a \in A, t \in [0, \infty). \tag{4}$$

Note that $\sigma \approxeq \sigma' \Rightarrow \sigma \approxeq_d \sigma' \Rightarrow \sigma(a) - \sigma'(a) \in \mathcal{K}_B(E), \quad a \in A.$

A representation $\sigma : A \to \mathcal{L}_B(E)$ is *strictly nuclear* if the map $x^*\sigma(-)x : A \to \mathcal{K}_B(E)$ is nuclear for any $x \in \mathcal{K}_B(E)$ [Sk]. A (unital) representation $\pi : A \to \mathcal{L}_B(E)$ is called *(unitally) absorbing* (respectively *nuclearly absorbing*) if $\pi \oplus \sigma \simeq \pi$ for any (unital) (respectively, strictly nuclear) representation $\sigma : A \to \mathcal{L}_B(F)$. If either A or B is nuclear, then any representation $\sigma : A \to \mathcal{L}_B(E)$ is strictly nuclear. In this case the notions of (unitally) nuclearly absorbing and (unitally) absorbing coincide. By [Kas1, Theorem 4](see [DE1, Proposition 2.18]), if $\rho : A \to \mathcal{L}(\mathcal{H})$ is a faithful unital representation with $\rho(A) \cap \mathcal{K}(\mathcal{H}) = \{0\}$, then by composing the inclusion $\mathcal{L}(\mathcal{H}) \hookrightarrow \mathcal{L}(\mathcal{H}_B)$ with ρ one obtains a strictly nuclear unitally nuclearly absorbing representation. It follows that the restriction of a unitally nuclearly absorbing representation $\sigma : A \to \mathcal{L}(\mathcal{H}_B)$ to a unital C*-subalgebra of A is unitally nuclearly absorbing (see [DE1, Proposition 2.19]).

The Cuntz picture of $KK(A, B)$ is described in terms of pairs of representations $(\varphi, \psi) : A \to \mathcal{L}(\mathcal{H}_B)$ satisfying $\varphi(a) - \psi(a) \in \mathcal{K}(\mathcal{H}_B)$ for $a \in A$. Such a pair is called a Cuntz pair. The set of Cuntz pairs is denoted by $\mathcal{E}(A, B)$. A homotopy of Cuntz pairs is a Cuntz pair $(\Phi, \Psi) \in \mathcal{E}(A, B[0, 1])$. The quotient of $\mathcal{E}(A, B)$ by homotopy equivalence is a group isomorphic to $KK(A, B)$ [Bl]. Let $\mathcal{E}_{nuc}(A, B)$ denote the set of Cuntz pairs consisting of strictly nuclear representations. Similarly, the group $\mathrm{KK}_{nuc}(A, B)$ of [Sk] is isomorphic to the quotient of $\mathcal{E}_{nuc}(A, B)$ by homotopy equivalence. The following result shows that after stabilization it suffices to work with just unitary homotopy equivalence.

Theorem 2.2 ([DE2]). *Let A be a (unital) separable C*-algebra and let B be a σ-unital C*-algebra. If $\varphi, \psi \in \mathcal{E}_{nuc}(A, B)$ is a Cuntz pair of (unital) representations then the following are equivalent:*

(i) $[\varphi, \psi] = 0$ *in* $\mathrm{KK}_{nuc}(A, B)$.
(ii) *There exists a (unital) strictly nuclear representation $\sigma : A \to \mathcal{L}(\mathcal{H}_B)$ with $\varphi \oplus \sigma \approxeq \psi \oplus \sigma$.*

(iii) *For any strictly nuclear (unitally) absorbing (unital) representation* $\gamma : A \to \mathcal{L}(\mathcal{H}_B)$*,* $\varphi \oplus \gamma_\infty \approxeq \psi \oplus \gamma_\infty$*, where* $\gamma_\infty = \gamma \oplus \gamma \oplus \cdots$*.*

Consequently $KK_{nuc}(A, B)$ can be described as the quotient of $\mathcal{E}_{nuc}(A, B)$ by the equivalence relation $(\varphi, \psi) \sim (\varphi', \psi')$ if and only if $\varphi \oplus \psi' \oplus \sigma \approxeq \psi \oplus \varphi' \oplus \sigma$ for some strictly nuclear representation $\sigma : A \to \mathcal{L}(\mathcal{H}_B)$.

It is useful to repeat the same construction with $\approxeq$ replaced by $\approxeq_d$.

Definition 2.3. We define

$$\widehat{\mathrm{KK}}_{nuc}(A, B) = \{ [\varphi, \psi]^\frown : (\varphi, \psi) \in \mathcal{E}_{nuc}(A, B)\}$$

where $[\varphi, \psi]^\frown = [\varphi', \psi']^\frown$ if and only if $\varphi \oplus \psi' \oplus \sigma \approxeq_d \psi \oplus \varphi' \oplus \sigma$ for some strictly nuclear representation $\sigma : A \to \mathcal{L}(\mathcal{H}_B)$.

Proposition 2.4. $\widehat{\mathrm{KK}}_{nuc}(A, B)$ *is an abelian group isomorphic to a quotient of* $\mathrm{KK}_{nuc}(A, B)$*. If* $\varphi, \psi \in \mathcal{E}_{nuc}(A, B)$ *is a Cuntz pair of (unital) representations then the following are equivalent:*

(i_d) $[\varphi, \psi]^\frown = 0$ *in* $\widehat{\mathrm{KK}}_{nuc}(A, B)$*.*

(ii_d) *There exists a (unital) strictly nuclear representation* $\sigma : A \to \mathcal{L}(\mathcal{H}_B)$ *with* $\varphi \oplus \sigma \approxeq_d \psi \oplus \sigma$*.*

(iii_d) *For any (unital) strictly nuclear (unitally) absorbing representation* $\gamma : A \to \mathcal{L}(\mathcal{H}_B)$*,* $\varphi \oplus \gamma \approxeq_d \psi \oplus \gamma$*.*

Proof. It is easy to see that $\approxeq_d$ has the following properties.

(A) If $\varphi \oplus \sigma \approxeq_d \psi \oplus \sigma$ and $\sigma \simeq \gamma$ then $\varphi \oplus \gamma \approxeq_d \psi \oplus \gamma$ (cf. [DE2, Lemma 3.4]).

(B) If $\varphi \approxeq_d \psi$, then $w\varphi w^* \approxeq_d w\psi w^*$ for any unitary $w \in \mathcal{L}(\mathcal{H}_B)$.

The addition on $\widehat{\mathrm{KK}}_{nuc}(A, B)$ is induced by the direct sum of Cuntz pairs. Let us check that the addition is well-defined. Suppose that $[\varphi, \psi]^\frown = [\varphi', \psi']^\frown$ and $[\alpha, \beta]^\frown = [\alpha', \beta']^\frown$. Then $\varphi \oplus \psi' \oplus \sigma \approxeq_d \psi \oplus \varphi' \oplus \sigma$ and $\alpha \oplus \beta' \oplus \gamma \approxeq_d \beta \oplus \alpha' \oplus \gamma$ for some strictly nuclear representations $\sigma, \gamma : A \to \mathcal{L}(\mathcal{H}_B)$. By taking direct sums, $\varphi \oplus \psi' \oplus \sigma \oplus \alpha \oplus \beta' \oplus \gamma \approxeq_d \psi \oplus \varphi' \oplus \sigma \oplus \beta \oplus \alpha' \oplus \gamma$. Using (B) we obtain $\varphi \oplus \alpha \oplus \psi' \oplus \beta' \oplus \sigma \oplus \gamma \approxeq_d \psi \oplus \beta \oplus \varphi' \oplus \alpha' \oplus \sigma \oplus \gamma$ hence $[\varphi \oplus \alpha, \psi \oplus \beta]^\frown = [\varphi' \oplus \alpha', \psi' \oplus \beta']^\frown$. Since $\varphi \approxeq \psi \Rightarrow \varphi \approxeq_d \psi$, the map $\mathrm{KK}_{nuc}(A, B) \to \widehat{\mathrm{KK}}_{nuc}(A, B)$ is a well-defined surjective morphism of semigroups. Since $\mathrm{KK}_{nuc}(A, B)$ is an abelian group, so is $\widehat{\mathrm{KK}}_{nuc}(A, B)$. In particular, the neutral element of $\widehat{\mathrm{KK}}_{nuc}(A, B)$ is given by the class of $[\sigma, \sigma]$ for some strictly nuclear representation $\sigma : A \to \mathcal{L}(\mathcal{H}_B)$. By using (A) and (B) one checks immediately the equivalence of (i_d), (ii_d) and (iii_d).

Finally, let us observe that the unital case follows from the following remark. Assume that A, φ and ψ are unital, $(\varphi, \psi) \in \mathcal{E}_{nuc}(A, B)$ and

$$\varphi \oplus \sigma \approxeq_d \psi \oplus \sigma \tag{5}$$

for some strictly nuclear representation $\sigma : A \to \mathcal{L}(\mathcal{H}_B)$. We claim that $\varphi \oplus \gamma \approxeq_d \psi \oplus \gamma$ for any unital unitally absorbing strictly nuclear representation $\gamma : A \to \mathcal{L}(\mathcal{H}_B)$. Indeed, let $E = \sigma(1)\mathcal{H}_B$ and let $\sigma' : A \to \mathcal{L}_B(E)$ be the corestriction of σ to E. If u_n is the sequence of unitaries implementing (5) and $p = 1_{\mathcal{H}_B} \oplus \sigma(1)$, then $[p, u_n] \to 0$. By functional calculus we find a sequence of v_n unitaries in $\mathbb{C}1_{\mathcal{H}_B \oplus E} + \mathcal{K}(\mathcal{H}_B \oplus E)$, satisfying $\|v_n - pu_n p\| \to 0$ and implementing $\varphi \oplus \sigma' \approxeq_d \psi \oplus \sigma'$. Consequently we have $\varphi \oplus \sigma' \oplus \gamma \approxeq_d \psi \oplus \sigma' \oplus \gamma$ and the claim follows from (A) since $\sigma' \oplus \gamma \simeq \gamma$. ☐

Let (a_n) be a sequence dense in the unit ball of A. If $\alpha, \beta : A \to \mathcal{L}(\mathcal{H}_B)$ are two representations we set $\mathrm{dist}(\alpha, \beta) = \sum_{n=1}^{\infty} 2^{-n} \|\alpha(a_n) - \beta(a_n)\|$. There is a natural topology on $\mathrm{KK}_{nuc}(A, B)$ (cf.

[BDF], [Sa], [Sch_1]) defined by the invariant pseudo-metric

$$d([\varphi,\psi],[\varphi',\psi']) = \inf_{u,\gamma}\{\,\mathrm{dist}(\varphi\oplus\psi'\oplus\gamma, u(\psi\oplus\varphi'\oplus\gamma)u^*)\,\}$$

where the infimum is taken after all unitaries $u \in \mathbb{C}1+\mathcal{K}(\mathcal{H}_B)$ and all strictly nuclear representations $\gamma : A \to \mathcal{L}(\mathcal{H}_B)$.

Remark 2.5. (a) It is easy to see that $d([\varphi,\psi],[\varphi',\psi']) = 0$ if and only if $[\varphi,\psi]\widehat{\ } = [\varphi',\psi']\widehat{\ }$. Therefore $[\varphi,\psi] \in \overline{\{0\}}$ in $\mathrm{KK}_{nuc}(A,B) \Leftrightarrow [\varphi,\psi]\widehat{\ } = 0$ in $\widehat{\mathrm{KK}}_{nuc}(A,B) \Leftrightarrow (\mathrm{ii}_d) \Leftrightarrow (\mathrm{iii}_d)$.

(b) $\widehat{\mathrm{KK}}_{nuc}(A,B) \cong \mathrm{KK}_{nuc}(A,B)/\overline{\{0\}}$ is the Hausdorff quotient of $\mathrm{KK}_{nuc}(A,B)$.

(c) Let A, B be unital C*-algebras with A separable quasidiagonal (relative to B) nuclear satisfying the UCT. Let $\varphi,\psi : A \to B$ be two unital $*$-homomorphisms. Then $[\varphi,\psi]\widehat{\ } = 0$ in $\widehat{\mathrm{KK}}(A,B)$ if and only if $[\varphi]-[\psi] \in \mathrm{Pext}(K_*(A), K_{*+1}(B))$ in $\mathrm{KK}(A,B)$ [D5, Theorem 5.1].

Proposition 2.6. *Let A, B be unital C*-algebras with A separable. Suppose that there exists an infinite-multiplicity sequence (χ_n) of unital $*$-homomorphisms from A to B such that for any nonzero element $a \in A$ the two-sided closed ideal of B generated by the set $\{\chi_1(a),\chi_2(a),\dots\}$ is equal to B. Then the representation $\chi = \chi_1\oplus\chi_2\oplus\cdots$, $\chi : A \to M(\mathcal{K}(\mathcal{H})\otimes B)$ is unitally nuclearly absorbing.*

Proof. This is very similar to the proof of [DE1, Theorem 2.22] which extends a result of Lin [L1]. As in [DE1, Proposition 2.19 and Lemma 2.21] it suffices to show that for any pure state φ of A, any $\mathcal{F} \subset A$ a finite subset and $\epsilon > 0$, there is a unit vector $\xi \in \mathcal{H}_B$ such that $\|\varphi(a)1_B - \langle\chi(a)\xi,\xi\rangle\| < \epsilon$, $a \in \mathcal{F}$. By applying the excision proposition of [AAP], one finds a pair of norm-one positive elements x, y such that $xy = yx = y$ and $\|\varphi(a)x^2 - xax\| < \epsilon$, $a \in \mathcal{F}$. By assumption, we find $b_1,\dots,b_m \in B$ such that $b_1^*\chi_1(y^2)b_1 + \cdots + b_m^*\chi_m(y^2)b_m = 1_B$. Then $\xi = (\chi_1(y)b_1,\dots,\chi_m(y)b_m)$ has the desired property. Indeed, $\|\varphi(a)1_B - \langle\chi(a)\xi,\xi\rangle\| = \|\sum_{i=1}^m b_i^*\chi_i(y)\chi_i(\varphi(a)x^2 - xax)\chi_i(y)b_i\| < \epsilon$.

For an alternate proof, if B is separable, one can apply the main result of [EKu]. Since each χ_n has infinite multiplicity in the given sequence, we may identify χ with $\chi' : A \to M(\mathcal{K}(\mathcal{H})\otimes\mathcal{K}(\mathcal{H})\otimes B)$, $\chi'(a) = 1\otimes\chi(a)$. Our assumption on (χ_n) clearly implies that for each non-zero element $a \in A$, the two-sided closed ideal of $\mathcal{K}(\mathcal{H})\otimes B$ generated by $\chi(a)(\mathcal{K}(\mathcal{H})\otimes B)$ is equal to $\mathcal{K}(\mathcal{H})\otimes B$. By [EKu, Theorems 6 and 17(iii)] it follows that the representation χ' (hence χ) is unitally nuclearly absorbing.

□

Let A, B be C*-algebras, let $\mathcal{F} \subset A$ be a finite subset and let $\epsilon > 0$. If $\varphi : A \to \mathcal{L}_B(E_\varphi)$ and $\psi : A \to \mathcal{L}_B(E_\psi)$ are two maps, we write $\varphi \underset{\mathcal{F},\epsilon}{\prec} \psi$ if there is an isometry $v \in \mathcal{L}_B(E_\varphi, E_\psi)$ such that $\|\varphi(a) - v^*\psi(a)v\| < \epsilon$ for all $a \in \mathcal{F}$. If v can be chosen to be a unitary, then we write $\varphi \underset{\mathcal{F},\epsilon}{\sim} \psi$. We write $\varphi \prec \psi$ ($\varphi \sim \psi$) if $\varphi \underset{\mathcal{F},\epsilon}{\prec} \psi$ (respectively $\varphi \underset{\mathcal{F},\epsilon}{\sim} \psi$) for all finite sets $\mathcal{F}$ and $\epsilon > 0$. Note that if $\varphi \underset{\mathcal{F},\epsilon_1}{\sim} \psi$ and $\psi \underset{\mathcal{F},\epsilon_2}{\sim} \gamma$, then $\varphi \underset{\mathcal{F},\epsilon_1+\epsilon_2}{\sim} \gamma$. Also, if two representations are approximately unitarily equivalent, $\sigma \simeq \sigma'$, then $\sigma \sim \sigma'$. Given a map $\varphi : A \to \mathcal{L}_B(E)$ we denote by φ_∞ the map $\oplus_{n=1}^\infty \varphi : A \to \mathcal{L}_B(\oplus_{n=1}^\infty E)$.

Lemma 2.7. *Let A be a C*-algebra, let $\mathcal{F} \subset A$ be a finite subset and let $\epsilon > 0$. There exist $\mathcal{G} \subset A$ a finite subset and $\delta > 0$ such that if $\varphi : A \to \mathcal{L}_B(E_\varphi)$ and $\psi : A \to \mathcal{L}_B(E_\psi)$ are selfadjoint maps with $\|\varphi(a^*a) - \varphi(a^*)\varphi(a)\| < \delta$, $\|\psi(a^*a) - \psi(a^*)\psi(a)\| < \delta$, $a \in \mathcal{G}$, then we have the following.*

(i) If $\varphi_\infty \underset{\mathcal{G},\delta}{\prec} \psi$*, then* $\varphi \oplus \psi \underset{\mathcal{F},\epsilon}{\sim} \psi$*.*

(ii) If $\varphi_\infty \underset{\mathcal{G},\delta}{\prec} \psi$ *and* $\psi_\infty \underset{\mathcal{G},\delta}{\prec} \varphi$*, then* $\varphi \underset{\mathcal{F},\epsilon}{\sim} \psi$*.*

Proof. This was proved in [D4] in the case $B = \mathbb{C}$. The same proof is valid in the general case considered here. □

Definition 2.8. Let A, B be unital C*-algebras with A separable. Let $\mathcal{F} \subset A$ be a finite subset and let $\epsilon > 0$. A unital $*$-homomorphism $\pi : A \to M_k(B)$ is called $(\mathcal{F},\epsilon)$*-admissible* if there is a unitally nuclearly absorbing representation $\sigma : A \to \mathcal{L}(\mathcal{H}_B)$, ($\mathcal{H}_B = B^k \oplus B^k \oplus \dots$) such that

$$\|\sigma(a) - \pi_\infty(a)\| < \epsilon \quad a \in \mathcal{F}. \tag{6}$$

Remark 2.9. (a) If π is $(\mathcal{F},\epsilon)$-admissible, then $\|\pi(a)\| \geq \|a\| - \epsilon$, $a \in \mathcal{F}$. Moreover, $\pi \oplus \gamma$ is $(\mathcal{F},\epsilon)$-admissible for any unital $*$-homomorphism $\gamma : A \to M_r(B)$.

(b) If $\gamma : A \to \mathcal{L}(\mathcal{H}_B)$ is any unitally nuclearly absorbing representation, then $\sigma \simeq \gamma$ hence $\|u\gamma(a)u^* - \pi_\infty(a)\| < \epsilon$, $a \in \mathcal{F}$, for some unitary $u \in \mathcal{L}(\mathcal{H}_B)$.

(c) Let $A' \subset A$ be a C*-subalgebra of A such that $1_A \in A'$. Let $\mathcal{F} \subset A'$ be a finite subset and let $\epsilon > 0$. If $\pi : A \to M_k(B)$ is $(\mathcal{F},\epsilon)$-admissible, then so is its restriction to A'. Indeed, as noticed earlier, the restriction to A' of a unitally nuclearly absorbing representation of A is a unitally nuclearly absorbing representation of A'.

Proposition 2.10. *Let A, B be unital C*-algebras with A separable and nuclearly embeddable. Suppose that there exists an infinite-multiplicity sequence* (χ_n) *of unital nuclear $*$-homomorphisms from A to B such that for any nonzero element* $a \in A$ *the two-sided closed ideal of B generated by the set* $\{\chi_1(a), \chi_2(a), \dots\}$ *is equal to B. Then for any* $\mathcal{F} \subset A$ *a finite set and any* $\epsilon > 0$ *there is a positive integer k such that* $\pi = \chi_1 \oplus \dots \oplus \chi_k : A \to M_k(B)$ *is* $(\mathcal{F},\epsilon)$*-admissible.*

Proof. Let $\theta : A \to \mathcal{L}(\mathcal{H})$ be a unital faithful representation with $\theta(A) \cap \mathcal{K}(\mathcal{H}) = \{0\}$. Since A is nuclearly embeddable, θ is nuclear. Denote $C = \mathcal{L}(\mathcal{H})$ and define $\gamma_0 : A \to B \otimes C$ by $\gamma_0(a) = 1_B \otimes \theta(a)$. Define $\chi_C : A \to M(\mathcal{K}(\mathcal{H}) \otimes B \otimes C)$ by $\chi_C(a) = \oplus_{n=1}^\infty \chi_n(a) \otimes 1_C$. By Proposition 2.6, the representation χ_C is unitally nuclearly absorbing. In particular it will absorb the nuclear $*$-homomorphism γ_0. Therefore $\gamma_0 \prec \gamma_0 \oplus \chi_C \sim \chi_C$, hence $\gamma_0 \prec \chi_C$. Let $\mathcal{G} \subset A$ and $\delta > 0$ be as in Lemma 2.7 corresponding to the given $\mathcal{F} \subset A$ and $\epsilon > 0$. Then we find an isometry $v \in \mathcal{L}_{B\otimes C}(B \otimes C, \mathcal{H} \otimes B \otimes C)$ such that $\|\gamma_0(a) - v^*\chi_C(a)v\| < \delta$ for $a \in \mathcal{G}$. After a small perturbation we may assume that the range of v is contained in $\mathbb{C}^k \otimes B \otimes C$ for some k. Then we can regard v as an isometry in $\mathcal{L}_{B\otimes C}(B \otimes C, \mathbb{C}^k \otimes B \otimes C)$ and $v^*\chi_C(a)v = v^*(\pi(a) \otimes 1_C)v$ where $\pi = \chi_1 \oplus \chi_2 \oplus \dots \oplus \chi_k$. Thus we obtain

$$\|\gamma_0(a) - v^*(\pi(a) \otimes 1_C)v\| < \delta, \quad a \in \mathcal{G}. \tag{7}$$

Let $\gamma : A \to M(B \otimes \mathcal{K}(\mathcal{H}))$ be defined as the composition of the inclusion map $B \otimes C = B \otimes \mathcal{L}(\mathcal{H}) \subset M(B \otimes \mathcal{K}(\mathcal{H}))$ with γ_0. Since v can be also viewed as an element of $M_k(B \otimes C) = M_k(B \otimes \mathcal{L}(\mathcal{H})) \subset M_k(M(B \otimes \mathcal{K}(\mathcal{H})))$, hence as an element (isometry) in $\mathcal{L}_B(B \otimes \mathcal{H}, \mathbb{C}^k \otimes B \otimes \mathcal{H})$, it follows from (7) that $\gamma \underset{\mathcal{G},\delta}{\prec} \pi \otimes 1_{\mathcal{L}(\mathcal{H})}$. Since $\pi \otimes 1_{\mathcal{L}(\mathcal{H})}$ is unitarily equivalent to π_∞ we obtain

$$\gamma \underset{\mathcal{G},\delta}{\prec} \pi_\infty. \tag{8}$$

On the other hand the representation $\gamma : A \to M(\mathcal{K}(\mathcal{H}) \otimes B)$ is unitally nuclearly absorbing by [Kas1, Theorem 4](see [DE1, Proposition 2.18]). Since each χ_n is nuclear, π_∞ is nuclear. Thus

$\pi_\infty \prec \pi_\infty \oplus \gamma \sim \gamma$, hence

$$\pi_\infty \prec \gamma. \tag{9}$$

By Lemma 2.7(ii), it follows from (8) and (9) that $\gamma \underset{\mathcal{F},\epsilon}{\sim} \pi_\infty$, hence $\|u\gamma(a)u^* - \pi_\infty(a)\| < \epsilon$, $a \in \mathcal{F}$, for some unitary $u \in M(\mathcal{K}(\mathcal{H}) \otimes B)$. We conclude the proof by setting $\sigma(a) = u\gamma(a)u^*$. ☐

If n is a positive integer and $\pi : A \to B$ is a $*$-homomorphism, then $n\pi : A \to M_n(B)$ will denote the $*$-homomorphism $\pi \oplus \cdots \oplus \pi$ (n-times). Let $\varphi, \psi : A \to \mathcal{K}(\mathcal{H}_B)$ be nuclear $*$-homomorphisms. We will write $[\varphi]^\frown$ for the class of the Cuntz pair $(\varphi, 0)$. Therefore $[\varphi, \psi]^\frown = [\varphi]^\frown - [\psi]^\frown$ in $\widehat{\mathrm{KK}}_{nuc}(A, B)$. The following proposition is crucial for our embedding result.

Proposition 2.11. *Let A, B be unital C^*-algebras with A separable. Let $\mathcal{F} \subset A$ be a finite subset and let $\epsilon > 0$. Then for any $(\mathcal{F}, \epsilon)$-admissible $*$-homomorphism $\pi : A \to M_k(B)$ and any two unital nuclear $*$-homomorphisms $\varphi, \psi : A \to M_m(B)$ with $[\varphi]^\frown = [\psi]^\frown$ in $\widehat{\mathrm{KK}}_{nuc}(A, B)$, there exist a positive integer N and a unitary $u \in M_{m+Nk}(B)$ satisfying*

$$\|u(\varphi(a) \oplus N\pi(a))u^* - \psi(a) \oplus N\pi(a)\| < 3\epsilon, \quad a \in \mathcal{F}.$$

Proof. Let $\mathcal{F}$, ϵ and π be as in the statement. Then π satisfies (6) for some unitally nuclearly absorbing representation $\sigma : A \to M(\mathcal{K}(\mathcal{H}) \otimes B)$. By applying Proposition 2.4 to φ, ψ and σ we find a unitary $v \in \mathbb{C}1 + \mathcal{K}_B(B^m \oplus \mathcal{H} \otimes B)$ such that

$$\|v(\varphi(a) \oplus \sigma(a))v^* - \psi(a) \oplus \sigma(a)\| < \epsilon, \quad a \in \mathcal{F}. \tag{10}$$

From (6) and (10) we then obtain

$$\|v(\varphi(a) \oplus \pi_\infty(a))v^* - \psi(a) \oplus \pi_\infty(a)\| < 3\epsilon, \quad a \in \mathcal{F}. \tag{11}$$

Let $\mathcal{H}_n = B^m \oplus B^k \oplus \cdots \oplus B^k \subset B^m \oplus \mathcal{H} \otimes \mathcal{B}$ (n copies of B^k) and let e_n denote the orthogonal projection of $B^m \oplus \mathcal{H} \otimes B$ onto $\mathcal{H}_n$. After a small perturbation of v we may assume that $v \in \mathbb{C}1 + \mathcal{K}_B(\mathcal{H}_N)$ for some large N. It is then clear that e_N commutes with v and with the images of $\varphi \oplus \pi_\infty$ and $\psi \oplus \pi_\infty$. Then $e_N(\varphi \oplus \pi_\infty)e_N = \varphi \oplus N\pi$, $e_N(\psi \oplus \pi_\infty)e_N = \psi \oplus N\pi$ and $u = e_N v e_N$ is a unitary in $\mathcal{L}_B(\mathcal{H}_N) \cong M_{m+Nk}(B)$. We finish the proof by compressing by e_N in (11).

☐

Proposition 2.12. *Let A be a separable unital quasidiagonal nuclearly embeddable C^*-algebra satisfying the UCT. Let B be a unital C^*-algebra such that B has bounded exponential length and $B \cong B \otimes \mathcal{U}$. Let $\mathcal{F} \subset A$ be a finite subset and let $\epsilon > 0$. There is a finitely generated subgroup $X \subset K_*(A)$ such that for any two unital nuclear $*$-homomorphisms $\varphi, \psi : A \to M_m(B)$ with $\varphi_*(x) = \psi_*(x)$, $x \in X$, and any $(\mathcal{F}, \epsilon)$-admissible $*$-homomorphism $\pi : A \to M_k(B)$, there exist a positive integer N and a unitary $u \in M_{m+Nk}(B)$ such that*

$$\|u(\varphi(a) \oplus N\pi(a))u^* - \psi(a) \oplus N\pi(a)\| < 3\epsilon, \quad a \in \mathcal{F}. \tag{12}$$

Proof. Let A, B, $\mathcal{F}$ and ϵ be as in the statement. Seeking a contradiction, suppose that there is no finitely generated subgroup X of $K_*(A)$ satisfying the conclusion of the proposition.

Since A is separable, $K_*(A)$ is countable. Therefore we can find an increasing sequence (X_n) of finitely generated subgroups of $K_*(A)$ whose union is equal to $K_*(A)$, two sequences of unital nuclear $*$-homomorphisms $\varphi_n, \psi_n : A \to M_{m(n)}(B)$ with $(\varphi_n)_*(x) = (\psi_n)_*(x)$, $x \in X_n$ and a sequence $\pi_n : A \to M_{k(n)}(B)$ of $(\mathcal{F}, \epsilon)$-admissible $*$-homomorphisms such that for all positive integers N and n

$$\inf_{v \in U_{m(n)+Nk(n)}(B)} \max_{a \in \mathcal{F}} \|v(\varphi_n(a) \oplus N\pi_n(a))v^* - \psi_n(a) \oplus N\pi_n(a)\| \geq 3\epsilon. \tag{13}$$

A contradiction will be obtained by showing that there is n such that for any $(\mathcal{F}, \epsilon)$-admissible $*$-homomorphism $\rho : B \to M_k(B)$ there exists a unitary $u \in \mathbb{C}1 + \mathcal{K}(\mathcal{H}) \otimes B$ with

$$\max_{a \in \mathcal{F}} \|u(\varphi_n(a) \oplus \rho_\infty(a))u^* - \psi_n(a) \oplus \rho_\infty(a)\| < 3\epsilon. \tag{14}$$

Indeed, by taking $\rho = \pi_n$, after compressing in (14) by a suitable projection as in the proof of Proposition 2.11, we contradict (13). Let $B_n = M_{m(n)}(B)$, $C = \prod B_n / \sum B_n$ and let $\Phi, \Psi : A \to C$ be the unital $*$-homomorphisms induced canonically by the sequences (φ_n) and (ψ_n). The maps Φ and Ψ are nuclear since A is nuclearly embeddable and φ_n, ϕ_n are nuclear [D1, Proposition 3.3]. We claim that

$$\Phi_* = \Psi_* : K_*(A) \to K_*(C). \tag{15}$$

Consider the commutative diagram whose rows are exact sequences.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_*(\sum B_n) & \longrightarrow & K_*(\prod B_n) & \longrightarrow & K_*(\prod B_n/\sum B_n) & \longrightarrow & 0 \\ & & \| & & \downarrow \nu_* & & \downarrow \dot{\nu}_* & & \\ 0 & \longrightarrow & \sum K_*(B_n) & \longrightarrow & \prod K_*(B_n) & \longrightarrow & \prod K_*(B_n)/\sum K_*(B_n) & \longrightarrow & 0 \end{array}$$

Note that if ν_* is injective, then so is $\dot{\nu}_*$. Since the union of (X_n) is equal to $K_*(A)$, and since $(\varphi_n)_*(x) = (\psi_n)_*(x)$ for $x \in X_n$, we have $\dot{\nu}_*\Phi_* = \dot{\nu}_*\Psi_*$. Therefore in order to prove (15) it suffices to prove that the canonical maps

$$\nu_i : K_i(\prod B_n) \to \prod K_i(B_n), \quad i = 0, 1 \tag{16}$$

are injective. Since B is simple and $B \cong B \otimes \mathcal{U}$, it follows from [Ro] that either B has stable rank one (hence cancellation of projections) or B is purely infinite. In either case it is easy to check that ν_0 is injective. As for the injectivity of ν_1, that follows from the assumption that B has bounded exponential length (see [EL] or [L1]).

Next we observe that $K_*(\prod B_n)$ and hence $K_*(C)$ is a divisible group since $B_n \cong B_n \otimes \mathcal{U}$. Since A satisfies the UCT [RS] it follows that $\mathrm{KK}(A, C) = Hom(K_*(A), K_*(C))$ and A is KK-equivalent to an abelian C*-algebra. Therefore A is K-nuclear and $\mathrm{KK}(A, C) \cong \mathrm{KK}_{nuc}(A, C)$ [Sk]. In conjunction with (15) this shows that $[\Phi] = [\Psi]$ in $\mathrm{KK}_{nuc}(A, C)$.

Let $\theta : A \to \mathcal{L}(\mathcal{H})$ be a unital faithful representation with $\theta(A) \cap \mathcal{K}(\mathcal{H}) = \{0\}$. Then $\theta \otimes 1_C : A \to M(\mathcal{K}(\mathcal{H}) \otimes C)$ is unitally nuclearly absorbing. By Theorem 2.2 there exists a unitary valued norm-continuous map $u : [0, \infty) \to \mathbb{C}1 + \mathcal{K}(\mathcal{H}) \otimes C$ such that $\lim_{t\to\infty} \|u_t(\Phi(a) \oplus \theta(a) \otimes 1_C)u_t^* - \Psi(a) \oplus \theta(a) \otimes 1_C\| = 0$, for all $a \in A$. From this we find a unitary $w \in \mathbb{C}1 + f\mathcal{K}(\mathcal{H})f \otimes C$ (with f a finite dimensional projection, $f \geq e_{11}$) such that

$$\|w(\Phi(a) \oplus \theta(a) \otimes 1_C)w^* - \Psi(a) \oplus \theta(a) \otimes 1_C\| < \epsilon/5, \quad a \in \mathcal{F}. \tag{17}$$

Since A is quasidiagonal, we find a projection $e \in \mathcal{K}(\mathcal{H})$, $e \geq f$, such that if $\theta'(a) = e\theta(a)e$ and $\theta''(a) = (1-e)\theta(a)(1-e)$, $a \in A$, then

$$\|\theta(a) - \theta'(a) - \theta''(a)\| < \epsilon/5, \quad a \in \mathcal{F}. \tag{18}$$

It is clear that $[w, e\otimes 1_C]=0$ since $e\geq f$ so that $v=(e\otimes 1_C)w(e\otimes 1_C)$ is a unitary in $e\mathcal{K}(\mathcal{H})e\otimes C\cong M_r(C)$, where r is the rank of e. From (17) and (18) we obtain

$$\|w(\Phi(a)\oplus(\theta'(a)+\theta''(a))\otimes 1_C)w^*-\Psi(a)\oplus(\theta'(a)+\theta''(a))\otimes 1_C\|<3\epsilon/5,\quad a\in\mathcal{F}. \tag{19}$$

After compressing by $e\otimes 1_C$ in (19) we obtain

$$\|v(\Phi(a)\oplus\theta'(a)\otimes 1_C)v^*-\Psi(a)\oplus\theta'(a)\otimes 1_C\|<3\epsilon/5,\quad a\in\mathcal{F}. \tag{20}$$

Note that θ' can be regarded as a map into $M_r(\mathbb{C})$, hence $\Phi\oplus\theta'\otimes 1_C, \Psi\oplus\theta'\otimes 1_C : A\to M_{r+1}(C)$. It its clear that $\theta'\otimes 1_C$ lifts to $(\theta'\otimes 1_{B_n}) : A\to M_r(\prod B_n)\cong\prod M_r(B_n)$. Let (v_n) be a unitary lifting of v in $\prod M_{r+1}(B_n)$. Then it follows from (20) that there is n such that

$$\|v_n(\varphi_n(a)\oplus\theta'(a)\otimes 1_{B_n})v_n^*-\psi_n(a)\oplus\theta'(a)\otimes 1_{B_n}\|<3\epsilon/5,\quad a\in\mathcal{F}. \tag{21}$$

If $z_n=v_n+(1-e)\otimes 1_{B_n}$, it follows from (21) and (18) that

$$\|z_n(\varphi_n(a)\oplus\theta(a)\otimes 1_{B_n})z_n^*-\psi_n(a)\oplus\theta(a)\otimes 1_{B_n}\|<\epsilon,\quad a\in\mathcal{F}. \tag{22}$$

Let $\rho : A\to M_k(B)$ be an $(\mathcal{F},\epsilon)$-admissible $*$-homomorphism. Since $\theta\otimes 1_{B_n}$ is unitally nuclearly absorbing, it follows from Remark 2.9(b) that there is a unitary $u_n\in M(\mathcal{K}(\mathcal{H})\otimes B_n)\cong M(\mathcal{K}(\mathcal{H})\otimes B)$ such that

$$\|u_n(\theta(a)\otimes 1_{B_n})u_n^*-\rho_\infty(a)\|<\epsilon,\quad a\in\mathcal{F}. \tag{23}$$

Define $u=(1_{B_n}\oplus u_n)z_n(1_{B_n}\oplus u_n^*)$. Then from (22) and (23)

$$\begin{aligned}\|u(\varphi_n(a)\oplus\rho_\infty(a))u^*-\psi_n(a)\oplus\rho_\infty(a)\| &= \|z_n(\varphi_n(a)\oplus u_n^*\rho_\infty(a)u_n)z_n^*-\psi_n(a)\oplus u_n^*\rho_\infty(a)u_n\|\\ &\leq 2\|u_n^*\rho_\infty(a)u_n-\theta(a)\otimes 1_{B_n}\|+\|z_n(\varphi_n(a)\oplus\theta(a)\otimes 1_{B_n})z_n^*-\psi_n(a)\oplus\theta(a)\otimes 1_{B_n}\|\\ &< 2\epsilon+\epsilon=3\epsilon.\end{aligned}$$

This proves (14) and concludes the proof. □

3. Embedding results

Theorem 3.1. *Let A, B be unital C^*-algebras with A separable and nuclearly embeddable. Suppose that there exist a sequence (A_n) of unital C^*-subalgebras of A (not necessarily nested) whose union is dense in A and a sequence $\chi_n : A\to B$ of unital nuclear $*$-homomorphisms satisfying the following conditions.*

(i) For any nonzero element $a\in A$ the two-sided closed ideal of B generated by the set $\{\chi_1(a),\chi_2(a),\dots\}$ is equal to B.

(ii) For each n, $\{[\chi_1|_{A_n}]^\frown\otimes 1_{\mathbb{Q}}, [\chi_2|_{A_n}]^\frown\otimes 1_{\mathbb{Q}},\dots\}$ generates a finite-dimensional subspace of $\widehat{\mathrm{KK}}_{nuc}(A_n, B)\otimes\mathbb{Q}$.

Then A embeds as a unital C^*-subalgebra of $B\otimes\mathcal{U}$ where $\mathcal{U}$ is the universal UHF algebra.

Proof. We may assume that the sequence (χ_n) has infinite multiplicity. Let (x_n) be a sequence dense in A and let $\epsilon_n=2^{-n}$. After passing to a subsequence of (A_n) we find for each n, $\mathcal{F}_n=\{a(n,1),a(n,2),\dots,a(n,n)\}\subset A_n$ such that $\|x_i-a(n,i)\|<\epsilon_n$ for $1\leq i\leq n$. Let $R(\chi)$ denote the set of unital $*$-homomorphisms from A to $M_k(B)$ which are unitarily equivalent to finite direct sums of the form $\chi_{i_1}\oplus\cdots\oplus\chi_{i_r}$, with k and r variable. By assumption, the image of $R(\chi)$ in $\widehat{\mathrm{KK}}_{nuc}(A_n,B)\otimes\mathbb{Q}$ generates a finite dimensional vector subspace H_n. Therefore, for each n, we find $\theta_{(n,1)},\dots,\theta_{(n,t(n))}\in R(\chi)$ such that $([\theta_{(n,i)}|_{A_n}]^\frown\otimes 1_{\mathbb{Q}})$, $1\leq i\leq t(n)$, is a system of generators of H_n. By Proposition 2.10, there exists a sequence (π_n) in $R(\chi)$ such that π_n is $(\mathcal{F}_n,\epsilon_n)$-admissible.

Define $\theta_n \in R(\chi)$ by $\theta_n = \theta_{(n,1)} \oplus \cdots \oplus \theta_{(n,t(n))} \oplus \pi_n$. Then $\theta_n|_{A_n}$ is $(\mathcal{F}_n, \epsilon_n)$-admissible since $\pi_n|_{A_n}$ is so by Remark 2.9.

We will construct inductively a sequence $(r(n))$ of positive integers and a sequence of unital nuclear $*$-homomorphisms $\gamma_n : A \to M_{k(n)}(B)$, $k(1) = r(1)$, $k(n) = k(n-1)r(n)$ for $n \geq 2$, such that

(i) γ_n is unitarily equivalent to $\theta_n \oplus \alpha_n$ for some $\alpha_n \in R(\chi)$.
(ii) $\|\gamma_{n+1}(a) - r(n)\gamma_n(a)\| < 3\epsilon_n$ for $a \in \mathcal{F}_n$.

First we set $\gamma_1 = \theta_1$, so that $r(1)$ is implicitly defined. Suppose that $\gamma_1, \ldots, \gamma_n$ and $r(1), \ldots, r(n)$ were constructed. By our choice of $\theta_{(n,i)}$, there are integers $k > 0$, (k_i) such that $k[\theta_{n+1}|_{A_n}]\hat{} = k_1[\theta_{(n,1)}|_{A_n}]\hat{} + \cdots + k_{t(n)}[\theta_{(n,t(n))}|_{A_n}]\hat{}$ in $\widehat{\mathrm{KK}}_{nuc}(A_n, B)$. Therefore if we define $\alpha'_{n+1} = (k-1)\theta_{n+1} \oplus (m-k_1)\theta_{(n,1)} \oplus \cdots \oplus (m - k_{t(n)})\theta_{(n,t(n))} \oplus m\pi_n \oplus m\alpha_n$, where $m > \max\{k_1, \ldots, k_{t(n)}\}$, then $[\theta_{n+1}|_{A_n} \oplus \alpha'_{n+1}|_{A_n}]\hat{} = m[\gamma_n|_{A_n}]\hat{}$ in $\widehat{\mathrm{KK}}_{nuc}(A_n, B)$. Since $\theta_n|_{A_n}$ is $(\mathcal{F}_n, \epsilon_n)$-admissible, so is $\gamma_n|_{A_n}$. By Proposition 2.11 there exist an integer N and a unitary $u \in M_{(m+N)k(n)}(B)$ such that

$$\|u(\theta_{n+1} \oplus \alpha'_{n+1} \oplus N\gamma_n)(a)\, u^* - (m\gamma_n \oplus N\gamma_n)(a)\| < 3\epsilon_n, \quad a \in \mathcal{F}_n.$$

Define $\alpha_{n+1} = \alpha'_{n+1} \oplus N\gamma_n$, $\gamma_{n+1} = u(\theta_{n+1} \oplus \alpha_{n+1})u^*$ and $r(n+1) = m + N$. Then it is clear that the conditions (i) and (ii) above are satisfied by $\gamma_1, \ldots, \gamma_{n+1}$ and $r(1), \ldots, r(n+1)$. From (ii) and the choice of $\mathcal{F}_n$, we have $\|\gamma_{n+1}(x_i) - r(n)\gamma_n(x_i)\| < 5\epsilon_n$ for $1 \leq i \leq n$. Let $\iota_n : M_{k(n)}(B) \hookrightarrow B \otimes \mathcal{U}$ be the canonical inclusion. Having the sequence γ_n available, we construct a unital embedding $\gamma : A \to \varinjlim M_{k(n)} \cong B$ by defining $\gamma(x)$, $x \in \{x_1, x_2, \ldots\}$, to be the limit of the Cauchy sequence $(\iota_n\gamma_n(x))$ and then extend to A by continuity. Note that γ is nuclear since all the maps in $R(\chi)$ are nuclear. Also $\|\gamma(x)\| = \|x\|$ since $\|\gamma_n(a)\| \geq \|a\| - \epsilon_n$ for $a \in \mathcal{F}_n$ (Remark 2.9(a)), hence $\|\gamma_n(x_i)\| \geq \|x_i\| - 3\epsilon_n$, $1 \leq i \leq n$, as $\|a(n,i) - x_i\| < \epsilon_n$. □

Theorem 3.2. *Let A, B be unital C*-algebras. Suppose that A is separable quasidiagonal nuclearly embeddable and satisfies the UCT and that $B \otimes \mathcal{U}$ has bounded exponential length. Suppose that there exists a sequence (χ_n) of unital nuclear $*$-homomorphisms from A to B such that for any nonzero element $a \in A$ the two-sided closed ideal of B generated by the set $\{\chi_1(a), \chi_2(a), \ldots\}$ is equal to B. Then A embeds as a unital C*-subalgebra of $B \otimes \mathcal{U}$.*

Proof. This is similar to the proof of Theorem 3.1, excepting that Proposition 2.12 replaces Proposition 2.11. Let $(\mathcal{F}_n)$ be a sequence of increasing finite subsets of A whose union is dense in A and let $\epsilon_n = 2^{-n}$. We may assume that the sequence (χ_n) has infinite multiplicity. Let (π_n) be a sequence in $R(\chi)$ such that π_n is $(\mathcal{F}_n, \epsilon_n)$-admissible. Let (X_n) be a sequence of finitely generated subgroups of $K_*(A)$ obtained by applying Proposition 2.12 to A, B, $\mathcal{F}_n$ and ϵ_n. For each n, the image H_n of the map $R(\chi) \to \mathrm{Hom}(K_*(A), K_*(B)) \to \mathrm{Hom}(X_n, K_*(B))$ is a finitely generated group. Therefore we find $\theta_{(n,1)}, \ldots, \theta_{(n,t(n))} \in R(\chi)$ such that $((\theta_{(n,i)})_*|_{X_n})$, $1 \leq i \leq t(n)$, is a system of generators of H_n. Define $\theta_n \in R(\chi)$ by $\theta_n = \theta_{(n,1)} \oplus \cdots \oplus \theta_{(n,t(n))} \oplus \pi_n$. Then $\theta_n|_{A_n}$ is $(\mathcal{F}_n, \epsilon_n)$-admissible since $\pi_n|_{A_n}$ is so.

As in the proof of Theorem 3.1 it suffices to construct inductively a sequence $(r(n))$ of positive integers and a sequence of unital nuclear $*$-homomorphisms $\gamma_n : A \to M_{k(n)}(B)$, $k(1) = r(1)$, $k(n) = k(n-1)r(n)$ for $n \geq 2$, such that

(i) γ_n is unitarily equivalent $\theta_n \oplus \alpha_n$ for some $\alpha_n \in R(\chi)$.
(ii) $\|\gamma_{n+1}(a) - r(n)\gamma_n(a)\| < 3\epsilon_n$ for $a \in \mathcal{F}_n$.

Set $\gamma_1 = \theta_1$ and suppose that $\gamma_1, \dots, \gamma_n$ and $r(1), \dots, r(n)$ were constructed. By our choice of $\theta_{(n,i)}$, there are integers (k_i) such that $(\theta_{n+1})_*(x) = k_1(\theta_{(n,1)})_*(x) + \cdots + k_{t(n)}(\theta_{(n,t(n))})_*(x), x \in X_n$. Therefore if $m > \max\{k_1, \dots, k_{t(n)}\}$, and we define $\alpha'_{n+1} = (m-k_1)\theta_{(n,1)} \oplus \cdots \oplus (m-k_{t(n)})\theta_{(n,t(n))} \oplus m\pi_n \oplus m\alpha_n$, then $(\theta_{n+1} \oplus \alpha_{n+1})_*(x) = m(\gamma_n)_*(x)$, for all $x \in X_n$. Since $\gamma_n|_{A_n}$ is $(\mathcal{F}_n, \epsilon_n)$-admissible, by Proposition 2.12 there exist a positive integers N and a unitary $u \in M_{(m+N)k(n)}(B)$ such that

$$\|u(\theta_{n+1}(a) \oplus \alpha'_{n+1} \oplus N\gamma_n(a))u^* - m\gamma_n(a) \oplus N\gamma_n(a)\| < 3\epsilon_n. \tag{24}$$

We conclude the proof by defining $\alpha_{n+1} = \alpha'_{n+1} \oplus N\gamma_n$, $\gamma_{n+1} = u(\theta_{n+1} \oplus \alpha_{n+1})u^*$ and $r(n+1) = m + N$. □

Proof of Theorem 1.1: Let B be any simple unital separable C*-algebra such that $B_n \subset B$ for all n and B satisfies the conditions from the statement. For instance $B = \bigotimes_{n=1}^{\infty} B_n$ (which is simple by [Ta, Corollary 4.21] since B_n are simple) or $B = B_1$ if all B_n are isomorphic.

The unital nuclear inclusion $A \subset \prod B_n$ defines a separating sequence of unital nuclear $*$-homomorphisms $\chi_n : A \to B$. Since B is simple, the condition (i) of Theorem 3.1 is satisfied. Condition (ii) is also satisfied since each vector space $\widehat{\mathrm{KK}}_{nuc}(A_i, B) \otimes \mathbb{Q}$ is finitely generated by assumption. We conclude the proof by applying Theorem 3.1.

The proof of the alternate form of Theorem 1.1 is similar, but one uses Theorem 3.2. □

Proof of Corollary 1.2: This follows from Theorem 1.1 since if A is RFD, then A is quasidiagonal and there is an infinite-multiplicity sequence of positive integers $k(n)$ such that A embeds in $\prod_{n=1}^{\infty} M_{k(n)}$. Moreover, the embedding $A \hookrightarrow \prod_{n=1}^{\infty} M_{k(n)}$ is nuclear since A is nuclearly embeddable (see for example [D1, 3.3]). To conclude the proof we observe that $\bigotimes_{n=1}^{\infty} M_{k(n)} \otimes \mathcal{U} \cong \mathcal{U}$. □

Proof of Corollary 1.5: If G is a second countable amenable locally compact MAP group, then $C^*(G)$ is nuclear [Co], it is residually finite dimensional by [BLS, Example 1.11(ii)] and satisfies the UCT by [Tu, Proposition 10.7]. Therefore the result follows from Corollary 1.2. □

Let us note that since $C^*(G)$ is nuclear, one can prove that $C^*(G)$ embeds in some simple AF algebra by arguing as above and applying the main result of [L2] instead of Corollary 1.2.

Corollary 3.3. *Let A be a separable unital nuclearly embeddable C*-algebra. Suppose that there exist a sequence (B_n) of separable simple unital AF algebras and a sequence $\chi_n : A \to B_n$ of unital $*$-homomorphisms separating the elements of A. If A satisfies the UCT, then A embeds as a unital C*-subalgebra of a simple AF algebra.*

Proof. This follows from Theorem 1.1 (2) since if B_n are simple AF algebras, then $B = \otimes_{n=1}^{\infty} B_n$ is a simple AF algebra and the exponential length of any AF algebra is equal to π. □

The following proposition is needed in the proof of Corollary 1.4. Related results have appeared in [BK] (for nuclear C*-algebras) and [D3], [Br3] (for exact C*-algebras).

Proposition 3.4. *Let $A \subset \mathcal{L}(\mathcal{H})$, $A \cap \mathcal{K}(\mathcal{H}) = \{0\}$, be a unital separable C*-algebra. If A is quasidiagonal, then there is an increasing sequence (D_n) of unital RFD C*-subalgebras of $A + \mathcal{K}(\mathcal{H})$ such that $A + \mathcal{K}(\mathcal{H}) = \overline{\bigcup_{n=1}^{\infty} D_n}$. If A is nuclearly embeddable, then D_n are nuclearly embeddable. If A satisfies the UCT, then we may arrange that D_n satisfy the UCT.*

Proof. Using the quasidiagonality of A, one finds as in [Ar], [Br3, Theorem 5.2] a sequence (e_n) of finite-dimensional mutually orthogonal projections with $\sum_{n=1}^{\infty} e_n = 1$, such that $\delta : A \to \mathcal{L}(\mathcal{H})$, $\delta(a) = \sum_{n=1}^{\infty} e_n a e_n$ is a well-defined unital completely positive map and $a - \delta(a) \in \mathcal{K}(\mathcal{H})$, $\|\delta(a)\| = \|a\|$ for all $a \in A$. We have $e_n \mathcal{L}(\mathcal{H}) e_n \cong \mathcal{L}(e_n \mathcal{H}) \cong M_{k(n)}(\mathbb{C})$ where $k(n) = \dim(e_n)$. By identifying

$\mathcal{H}$ with $\oplus_{n=1}^{\infty}\mathbb{C}^{k(n)}$, we have embeddings $\prod_{n=1}^{\infty} M_{k(n)} \subset \mathcal{L}(\mathcal{H})$ and $\sum_{n=1}^{\infty} M_{k(n)} \subset \mathcal{K}(\mathcal{H})$. Let $D = \delta(A) + \sum_{n=1}^{\infty} M_{k(n)} = C^*(\delta(A)) + \sum_{n=1}^{\infty} M_{k(n)}$. Then D is a unital RFD C*-algebra and $D + \mathcal{K}(\mathcal{H}) = A + \mathcal{K}(\mathcal{H})$.

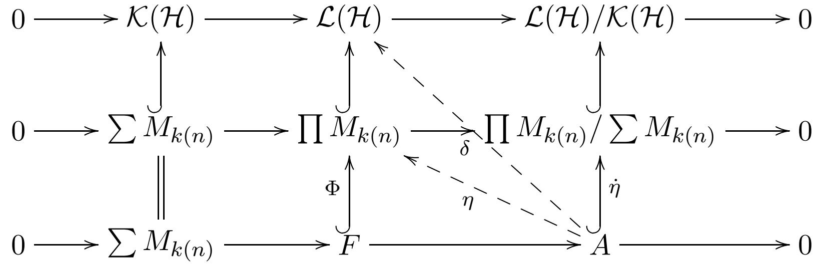
Let $E_n = e_1 \oplus \cdots \oplus e_n$ and $K(n) = k_1 + \cdots + k(n)$. Then $E_n\mathcal{L}(\mathcal{H})E_n \cong \mathcal{L}(E_n\mathcal{H}) \cong M_{K(n)}(\mathbb{C})$. We define $D_n = D + M_{K(n)}(\mathbb{C}) = D + E_n\mathcal{L}(\mathcal{H})E_n$ and note that $D_n \subset D_{n+1}$ and $D + \mathcal{K}(\mathcal{H}) = \overline{\bigcup_{n=1}^{\infty} D_n}$. Let $\sigma_n : D \to M_{k(n)}(\mathbb{C})$ be given by $\sigma_n(x) = e_n x e_n$. It is then clear that $D \cong \{\oplus_{n=1}^{\infty}\sigma_n(x) \ : \ x \in D\}$ and

$$D_n \cong M_{K(n)}(\mathbb{C}) \oplus D'_n \tag{25}$$

where $D'_n = \{\oplus_{i=1}^{\infty}\sigma_{n+i}(x) \ : \ x \in D\}$. Note that D'_n is a quotient of D by a finite-dimensional ideal J_n. Since any finite-dimensional C*-algebra is unital, D'_n is a direct summand in D.

$$D \cong J_n \oplus D'_n. \tag{26}$$

In particular D'_n is RFD. Consider the commutative diagram with exact rows



where $\eta(a) = (e_n a e_n)_{n=1}^{\infty}$ and the bottom row is the pullback of the middle row. If A is nuclearly embeddable, then η is nuclear and F is nuclearly embeddable as proved in [D4, Lemma 3.1]. Since $D \cong \Phi(F)$, D is nuclearly embeddable. It follows from (25) and (26) that D_n is nuclearly embeddable. Suppose now that A satisfies the UCT. By [Sk, Prop. 5.3] a separable C*-algebra F satisfies the UCT if and only if $\mathrm{KK}(F, B) = 0$ for any σ-unital C*-algebra B with $K_*(B) = 0$. From this and the KK-theory exact sequence associated with semisplit exact sequence $0 \to \sum M_{k(n)} \to F \to A \to 0$, we see that $D \cong \Phi(F)$ satisfies the UCT if A does so. It follows from (25) and (26) that D_n satisfies the UCT. □

4. Embeddings of group C*-algebras

The following proposition and its proof was inspired by [Be, Proposition 1].

Proposition 4.1. *Let Γ be a discrete countable amenable group. Let (B_n) be sequence of unital C*-algebras and let (ω_n) be an infinite-multiplicity sequence of group homomorphisms $\omega_n : \Gamma \to U(B_n)$ separating the points of Γ. Then $C^*(\Gamma)$ embeds unitally in $\prod_{n=1}^{\infty} C_n$ where $C_n = \bigotimes_{k=1}^{n} M_2 \otimes B_k \otimes B_k$.*

Proof. By assumption, the infinite-multiplicity sequence (ω_n) separates the points of Γ. Therefore there is an injective map $s \mapsto n(s)$ from Γ to $\mathbb{N}$ such that $\omega_{n(s)}(s) \neq 1$ for $s \in \Gamma \setminus \{e\}$. Define $\mu_n : \Gamma \to U(M_2 \otimes B_n \otimes B_n)$ by $\mu_n(s) = \begin{pmatrix} \omega_n(s) \otimes 1 & 0 \\ 0 & \omega_n(s) \otimes \omega_n(s) \end{pmatrix}$. We regard $M_2 \otimes B_n \otimes B_n$ as acting on a Hilbert space $\mathcal{H}_n$ and denote by $\pi_n : C_n \to \mathcal{L}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n)$ the corresponding tensor product representation. The spectrum of $\mu_{n(s)}(s)$ contains at least two points since $\omega_{n(s)}(s) \neq 1$. Using the spectral theorem, we find a sequence a sequence $\xi_n \in \mathcal{H}_n$, $\|\xi_n\| = 1$, so that if $\varphi_n(s) = \langle \mu_n(s)\xi_n, \xi_n \rangle$, then $|\varphi_{n(s)}(s)| < 1$, for all $s \in \Gamma$, $s \in \Gamma \setminus \{e\}$. Define $\chi_n : \Gamma \to U(C_n)$ by $\chi_n(s) = \mu_1(s) \otimes \cdots \otimes \mu_n(s)$. Then $\phi_n(s) = \varphi_1(s) \cdots \varphi_n(s) = \langle \pi_n \chi_n(s) \xi_1 \otimes \cdots \otimes \xi_n, \xi_1 \otimes \cdots \otimes \xi_n \rangle$ is a positive definite function associated with the representation $\pi_n \chi_n : \Gamma \to \mathcal{L}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n)$. Let δ_e be the Dirac function at the unit of Γ. Then $\lim_{n\to\infty} \phi_n(s) = \delta_e(s)$ for all $s \in \Gamma$, since $\phi_n(e) = 1$ and $|\varphi_{n(s)}(s)| < 1$ for $s \neq e$ and the sequence (φ_n) has infinite multiplicity. Since the positive definite function δ_e corresponds to a cyclic vector of the left regular representation $\lambda_\Gamma : \Gamma \to \mathcal{L}(\ell^2(\Gamma))$, it follows by [Dix, 18.1.4] that λ_Γ is weakly contained in $\{\pi_n \chi_n : n \in \mathbb{N}\}$. Thus if $\widehat{\lambda}_\Gamma : C^*(\Gamma) \to \mathcal{L}(\ell^2(\Gamma))$ and $\widehat{\chi}_n : C^*(\Gamma) \to C_n$ denote the extensions of λ_Γ and χ_n to $C^*(\Gamma)$, then by [Dix, 3.3.4]

$$\ker \widehat{\lambda}_\Gamma \supset \bigcap_{n=1}^{\infty} \ker \pi_n \widehat{\chi}_n = \bigcap_{n=1}^{\infty} \ker \widehat{\chi}_n.$$

Since Γ is amenable, $\ker \widehat{\lambda}_\Gamma = \{0\}$, hence the unital $*$-homomorphism $\prod_{n=1}^\infty \widehat{\chi}_n : C^*(\Gamma) \to \prod_{n=1}^\infty C_n$ is injective. □

Proof of Theorem 1.6: Without loss of generality we may assume that the sequence of homomorphisms $\omega_n : \Gamma \hookrightarrow \prod_{n=1}^\infty U(B_n) \to U(B_n)$ has infinite multiplicity. By Proposition 4.1, $C^*(\Gamma) \hookrightarrow \prod_{n=1}^\infty C_n$ where $C_n = \bigotimes_{k=1}^n M_2 \otimes B_k \otimes B_k$. In particular $C^*(\Gamma)$ is quasidiagonal since all the $B_n's$ are so. The C*-algebras C_n are simple by [Ta, Corollary 4.21]. $C^*(\Gamma)$ is nuclear as Γ is amenable [La] and it satisfies the UCT by [Tu]. Finally $(\bigotimes_{n=1}^\infty C_n) \otimes \mathcal{U} \cong (\bigotimes_{n=1}^\infty B_n) \otimes \mathcal{U}$ has bounded exponential length by assumption. We conclude the proof by applying Theorem 1.1.

The proof of the alternate form of Theorem 1.6 is similar and follows from the alternate form of Theorem 1.1. Indeed, letting $C = \bigotimes_{n=1}^\infty C_n$, and using the UCT, we have that $\mathrm{KK}(C^*(\Gamma_i), C \otimes \mathcal{U})$ is isomorphic to $\mathrm{Hom}(K_*(C^*(\Gamma_i)), K_*(C) \otimes \mathbb{Q})$ which is finitely generated since $K_*(C^*(\Gamma_i)) \otimes \mathbb{Q}$ is so by assumption. □

Lemma 4.2. *Let Γ be a discrete group acting by automorphisms on a unital C*-algebra A. Suppose that A has a sequence of Γ-invariant two-sided closed ideals (I_n) with $\bigcap_{n=1}^\infty I_n = \{0\}$. Then $A \times_r \Gamma$ embeds unitally in $\prod_{n=1}^\infty (A/I_n) \times_r \Gamma$.*

Proof. This is similar to the proof of [To, Theorem 4.1.10]. The map $\ell^1(\Gamma, A) \to A$, $(a_s)_{s\in\Gamma} \mapsto a_e$ extends to a faithful conditional expectation $E_A : A \times_r \Gamma \to A$ by [ZM, 4.12]. Consider the commutative diagram

$$\begin{array}{ccc} A \times_r \Gamma & \xrightarrow{\pi_n} & A/I_n \times_r \Gamma \\ \big\downarrow E_A & & \big\downarrow E_{A/I_n} \\ A & \longrightarrow & A/I_n \end{array}$$

We claim that the map $\prod \pi_n : A \times_r \Gamma \to \prod A/I_n \times_r \Gamma$ is injective. Indeed, let $x \in A \times_r \Gamma$, $x \geq 0$, be such that $\pi_n(x) = 0$ for all n. From the commutative diagram, we obtain that $E_A(x) \in I_n$ for all n hence $E_A(x) = 0$ since $\bigcap_{n=1}^\infty I_n = \{0\}$ by assumption. Therefore $x = 0$ since E_A is faithful. □

Corollary 4.3. *Let Γ be a discrete countable amenable group which is isomorphic to a subgroup of a countable product of unitary groups of simple unital separable AF algebras. Suppose that Γ acts on a compact metrisable space such that the points with finite orbits are dense in X. Then $C(X) \times_r \Gamma$ embeds in a unital simple separable AF algebra.*

Proof. Since Γ is amenable so is any of its subgroups H and $C^*_r(H) \cong C^*(H) \subset C^*(\Gamma) \cong C^*_r(\Gamma)$. By assumption there is a dense sequence (x_n) of points of X such that each isotropy group $\Gamma_{x_n} = \{s \in \Gamma : s \cdot x_n = x_n\}$ has finite index in Γ, $[\Gamma : \Gamma_{x_n}] = m(n) < \infty$. Let I_n denote the ideal of $C(X)$ consisting of all functions vanishing on the orbit X_n of x_n. Then $A/I_n \times_r \Gamma \cong C(X_n) \times_r \Gamma \cong C^*_r(\Gamma_{x_n}) \otimes M_{m(n)} \subset C^*_r(\Gamma) \otimes M_{m(n)}$. We have $\bigcap_{n=1}^{\infty} I_n = \{0\}$ since (x_n) is dense in X. Therefore $C(X) \times_r \Gamma$ embeds unitally in $\prod_{n=1}^{\infty} C^*_r(\Gamma) \otimes M_{m(n)}$ by Lemma 4. By Theorem 1.6, $C^*_r(\Gamma) \otimes M_{m(n)}$ embeds unitally in a simple unital AF algebra B, hence $C(X) \times_r \Gamma \subset \prod_{n=1}^{\infty} B$. The groupoid $X \times \Gamma$ is amenable, hence $C(X) \times_r \Gamma$ satisfies the UCT by [Tu, Proposition 10.7]. We conclude the proof by applying Theorem 1.6. □

Note that the above corollary applies to actions of countable discrete amenable residually finite groups (including $\mathbb{Z}^n$), provided that they have dense sets of points with finite orbits.

Proof of Corollary 1.7: This is an immediate consequence of Theorem 1.6 and Corollary 4.3. □

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