

ALEXANDER DUALITY AND SERRE'S PROPERTY (S_i) FOR SQUARE-FREE MONOMIAL IDEALS

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ABSTRACT. In this note, we study Serre's property (S_i) , and its relation to Alexander duality for monomial ideals in a polynomial ring over a field. We describe ideals that define the non-Cohen-Macaulay- and the non- (S_i) -loci of finitely generated modules over regular rings, and show that minimal prime ideals in these loci are homogeneous when M is graded. We show that a square-free monomial ideal has property (S_i) if and only if its Alexander dual has a linear resolution up to homological degree $i - 1$. We prove that for square-free monomial ideals, having property (S_2) is equivalent to being locally connected in codimension 1.

1. MAIN RESULTS

Let \mathbb{k} be a field and $R = \mathbb{k}[x_1, \dots, x_n]$ a polynomial ring in n variables. The *standard grading* on R is the grading by \mathbb{N} , obtained by setting $\deg x_l = 1$ for all $1 \leq l \leq n$. We can also make R *multigraded*, *i.e.*, graded by \mathbb{N}^n , with $\deg x_l = \mathbf{e}_l, 1 \leq l \leq n$, \mathbf{e}_l being the l th standard basis vector for \mathbb{N}^n . For any finitely generated R -module M , we say that M *satisfies Serre's property (S_i)* if for all $\mathfrak{p} \in \text{Spec } R$, $\text{depth } M_{\mathfrak{p}} \geq \min\{i, \dim M_{\mathfrak{p}}\}$ ¹. We adopt the convention that the zero module has property (S_i) for all i . The *(S_i) -locus of M* is the set $\mathcal{U}_{(S_i)}(M) := \{\mathfrak{p} \in \text{Spec } R : M_{\mathfrak{p}} \text{ has property } (S_i)\}$. We know from [EGA, IV, 6.11.2] that this is an open subset of $\text{Spec } R$. In Proposition 9, we describe an ideal that defines the complement of $\mathcal{U}_{(S_i)}(M)$, following which (Discussion 11) we show that if M is graded, either by \mathbb{N} or by \mathbb{N}^n , then so are the minimal prime ideals in this closed set.

For any homogeneous ideal $I \subseteq R$, we say that I *satisfies property $(N_{c,i})$* (after [EGHP05]) if all the minimal generators of I have degree c and a minimal graded free resolution of I is linear up to homological degree $i - 1$. This definition is independent of the choice of the resolution, because I satisfies property $(N_{c,i})$ if and only if $\text{Tor}_l^R(\mathbb{k}, I)_j = 0$ for all $0 \leq l \leq i - 1$ and for all $j \neq l + c$. We now relate the properties (S_i) and $(N_{c,i})$:

Theorem 1. *Let $I \subseteq R$ be a square-free monomial ideal with $\text{ht } I = c$. Then for $i > 1$, the following are equivalent:*

- (a) R/I satisfies property (S_i) .
- (b) The Alexander dual I^* satisfies $(N_{c,i})$.

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¹This definition follows [EGA, IV, 5.7.2] and [BH93, Section 2.1]. There is another definition of Serre's condition (S_i) , used in [EG85, Section 0.B]: a module M is said to satisfy Serre's condition (S_i) if $\text{depth } M_{\mathfrak{p}} \geq \min\{i, \dim R_{\mathfrak{p}}\}$, for all $\mathfrak{p} \in \text{Spec } R$.

Remark 2. The Alexander dual of a square-free monomial ideal I , minimally generated by monomials f_1, \dots, f_s , is the (square-free monomial) ideal $\cap_{i=1}^s \mathfrak{p}_{f_i}$, where for any square-free monomial $f = x_{l_1} \cdots x_{l_j}$, we set $\mathfrak{p}_f := (x_{l_1}, \dots, x_{l_j})$. See [MS05] for more on Alexander duality, and, also for any unexplained terminology.

Remark 3. The motivation for Theorem 1 is the result of Eagon-Reiner [ER98] (see also [MS05, Theorem 5.56]) that R/I is Cohen-Macaulay if and only if I^* has a linear free resolution. We have that R/I is Cohen-Macaulay if and only if R/I satisfies property (S_i) for all i . By Theorem 1, this is equivalent to I^* having property $(N_{c,i})$ for all i .

Remark 4. Terai [Ter99] (see Proposition 13 below) gave a generalization of the Eagon-Reiner theorem; we require this in our proof of Theorem 1. For two other results generalizing the Eagon-Reiner theorem, see Herzog-Hibi [HH99, Theorem 2.1(a)] an Herzog-Hibi-Zheng [HHZ04, Theorem 1.2(c)].

Remark 5. We can extend the statement to include the case $i = 1$ by replacing the statement (a) by “ R/I satisfies property (S_i) and I is unmixed” (*i.e.*, for all the associated primes \mathfrak{p} of R/I , $\dim R/\mathfrak{p}$ is independent of \mathfrak{p}). Since R/I is reduced, it always satisfies property (S_1) . Hence if I is unmixed, then I^* is generated by monomials of degree c ; this is property $(N_{c,1})$ for I^* . For larger i , the hypothesis that I is unmixed becomes superfluous: for any ideal I , not necessarily homogeneous, if R/I satisfies property (S_2) , then I is unmixed [EGA, IV, 5.10.9].

For a commutative ring A , we say that $\text{Spec } A$ is *connected in codimension k* , if for all ideal $\mathfrak{a} \subseteq A$ with $\text{ht } \mathfrak{a} > k$, $\text{Spec } A \setminus \{\mathfrak{p} \in \text{Spec } A : \mathfrak{a} \subseteq \mathfrak{p}\}$ is connected, and that A is *locally connected in codimension k* if $A_{\mathfrak{p}}$ is connected in codimension k for all $\mathfrak{p} \in \text{Spec } A$. It is known [Har62, Corollary 2.4] that for any ideal I , not necessarily homogeneous, if R/I satisfies property (S_2) , then $\text{Spec } R/I$ is locally connected in codimension 1. For square-free monomial ideals, we prove the converse, giving the following equivalence:

Theorem 6. *Let $R = \mathbb{k}[x_1, \dots, x_n]$ be a polynomial ring in n variables and let $I \subseteq R$ be a square-free monomial ideal. Then $\text{Spec } R/I$ is locally connected in codimension 1 if and only if R/I satisfies property (S_2) .*

2. FREE RESOLUTIONS AND THE LOCUS OF NON- (S_i) POINTS

Many results in this section are part of folklore. We take R to be an arbitrary regular domain, and M a finitely generated R -module with a finite free resolution

$$\mathbb{F}_{\bullet} : \quad 0 \longrightarrow F_p \xrightarrow{\phi_p} F_{p-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0.$$

Let $c = \text{codim } M$. For $1 \leq l \leq p$, set $r_l := \sum_{j=l}^p (-1)^{j-l} \text{rk } F_j$ and $I_l := \sqrt{I_{r_l}(\phi_l)}$, where, for a map ϕ of free modules of finite rank, and a natural number t , $I_t(\phi)$ is ideal generated by the $t \times t$ minors of ϕ and $\sqrt{}$ denotes taking the radical of an ideal.

Remark 7. Since R is a domain, M has a well-defined rank. We apply [BE73, Lemma 1] to conclude that M is projective if and only if $I_1 = R$. We see immediately that the exact sequence $(0 \longrightarrow \text{Im } \phi_l \longrightarrow F_{l-1} \longrightarrow \text{coker } \phi_l \longrightarrow 0) \otimes_R R_{\mathfrak{p}}$ splits — we say that $\phi_l \otimes_R R_{\mathfrak{p}}$ *splits* if this happens — if and only if $I_l \not\subseteq \mathfrak{p}$. If $\phi_l \otimes_R R_{\mathfrak{p}}$ splits, then so does every $\phi_{l'} \otimes_R R_{\mathfrak{p}}$ for $l' \geq l$. Hence $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_p$.

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Additionally, if R is local, with maximal ideal \mathfrak{m} , and M is not free, then $\text{pd } M = \max\{l : 1 \leq l \leq p \text{ and } I_l \subseteq \mathfrak{m}\}$.

First we determine the Cohen-Macaulay locus of M , which is an open subset of $\text{Spec } R$; see [EGA, IV, 6.11.3]. Let

$$(1) \quad J_{CM}(M) := \bigcap_{k=c+1}^p \left(I_k + \bigcap_{\substack{\mathfrak{q} \in \min M, \\ \text{ht } \mathfrak{q} < k}} \mathfrak{q} \right),$$

taking $J_{CM}(M) = R$ if the intersection is empty.

Proposition 8. *For all $\mathfrak{p} \in \text{Spec } R$, $M_{\mathfrak{p}}$ is Cohen-Macaulay if and only if $J_{CM}(M) \not\subseteq \mathfrak{p}$.*

Proof. Let $l = \text{codim } M_{\mathfrak{p}} + 1$. First, $\left(I_k + \bigcap_{\substack{\mathfrak{q} \in \min M, \\ \text{ht } \mathfrak{q} < k}} \mathfrak{q} \right) \not\subseteq \mathfrak{p}$ for all $k < l$; otherwise, we would get an ideal $\mathfrak{q} \subseteq \mathfrak{p}$ with $\mathfrak{q} \in \min M$ and $\text{ht } \mathfrak{q} < \text{codim } M_{\mathfrak{p}}$, which is a contradiction. We now see that $M_{\mathfrak{p}}$ is Cohen-Macaulay if and only if $\text{pd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \text{codim } M_{\mathfrak{p}}$, or, equivalently (by Remark 7), $I_l \not\subseteq \mathfrak{p}$, or, equivalently (by Remark 7, again), $\left(I_k + \bigcap_{\substack{\mathfrak{q} \in \min M, \\ \text{ht } \mathfrak{q} < k}} \mathfrak{q} \right) \not\subseteq \mathfrak{p}$ for all $k \geq l$, or, equivalently (by above), $J_{CM}(M) \not\subseteq \mathfrak{p}$. □

In order to determine the (S_i) -locus of M , we first define $\Lambda_i = \Lambda_i(M)$ to be the set of all $\mathfrak{q} \in \text{Spec } R$ such that \mathfrak{q} is minimal over $I_l + J_{CM}(M)$ for some $l > \text{ht } \mathfrak{q} - i$. Note that Λ_i is finite. Now let $J_{(S_i)}(M) = \bigcap_{\mathfrak{q} \in \Lambda_i} \mathfrak{q}$, taking $J_{(S_i)}(M) = R$ if $\Lambda_i = \emptyset$.

Proposition 9. *For all $\mathfrak{p} \in \text{Spec } R$, $\mathfrak{p} \in \mathcal{U}_{(S_i)}(M)$ if and only if $J_{(S_i)}(M) \not\subseteq \mathfrak{p}$.*

Proof. Let $\mathfrak{p} \in \text{Spec } R$ and $\Lambda_i \cap \mathfrak{p} := \{\mathfrak{q} \in \Lambda_i : \mathfrak{q} \subseteq \mathfrak{p}\}$. Since $J_{(S_i)}(M) \not\subseteq \mathfrak{p}$ if and only if $\Lambda_i \cap \mathfrak{p} = \emptyset$, we need to show that $\mathfrak{p} \in \mathcal{U}_{(S_i)}(M)$ if and only if $\Lambda_i \cap \mathfrak{p} = \emptyset$.

Let $\mathfrak{q} \in \Lambda_i \cap \mathfrak{p}$. Let $l > \text{ht } \mathfrak{q} - i$ be such that \mathfrak{q} is minimal over $I_l + J_{CM}(M)$. We apply Remark 7 to the regular local ring $(R_{\mathfrak{q}}, \mathfrak{q}R_{\mathfrak{q}})$ to conclude that $\text{pd}_{R_{\mathfrak{q}}} M_{\mathfrak{q}} > \dim R_{\mathfrak{q}} - i$, and, by the Auslander-Buchsbaum formula, that $\text{depth } R_{\mathfrak{q}} < i$. Since $J_{CM}(M) \subseteq \mathfrak{q}$, $M_{\mathfrak{q}}$ is not Cohen-Macaulay. Hence $M_{\mathfrak{q}}$ does not have property (S_i) , so $\mathfrak{p} \notin \mathcal{U}_{(S_i)}(M)$.

Conversely, if $\mathfrak{p} \notin \mathcal{U}_{(S_i)}(M)$, then $\mathfrak{q} \subseteq \mathfrak{p}$ be such that $\text{depth } M_{\mathfrak{q}} < \min\{i, \dim M_{\mathfrak{q}}\}$. Then $M_{\mathfrak{q}}$ is not Cohen-Macaulay, *i.e.*, $J_{CM}(M) \subseteq \mathfrak{q}$, and $\text{pd}_{R_{\mathfrak{q}}} M_{\mathfrak{q}} > \dim R_{\mathfrak{q}} - i$. By Remark 7, there exists $l > \text{ht } \mathfrak{q} - i$ such that $I_l \subseteq \mathfrak{q}$. Let \mathfrak{q}' be minimal such that $I_l + J_{CM}(M) \subseteq \mathfrak{q}' \subseteq \mathfrak{q}$. Since \mathfrak{q}' is minimal over $I_l + J_{CM}(M)$ and $l > \text{ht } \mathfrak{q}' - i$, we see that $\mathfrak{q}' \in \Lambda_i \cap \mathfrak{p}$. □

Remark 10. Suppose that $\text{ht } \mathfrak{p} = c$ for all $\mathfrak{p} \in \min M$, *i.e.*, that $\text{Ann } M$ is unmixed. Then $J_{CM}(M) = I_{c+1} + \sqrt{\text{Ann } M}$. If $M = R/I$ for some radical ideal I , then $r_1 = 1$ and $I_1 = I$, so we get $J_{CM}(R/I) = I_{c+1}$. Hence Λ_i consists of those primes \mathfrak{q} minimal over I_l for some $l \geq c + 1$ with $\text{ht } \mathfrak{q} < l - i$.

Discussion 11. Let $R = \mathbb{k}[x_1, \dots, x_n]$, taken with standard grading, and M a finitely generated graded R -module. Let \mathbb{F}_{\bullet} be a *graded* free resolution of M , with maps of degree 0. Then the $I_{r_i}(\phi_l)$ are homogeneous: to show this, it is

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enough to show that if F and G are graded free modules of same finite rank and $\phi : F \rightarrow G$ is a map of degree 0, then $\det \phi$ is homogeneous. Indeed, giving bases f_1, \dots, f_r for F and g_1, \dots, g_r for G , we can write $\phi = [a_{ij}]$. If $a_{ij} \neq 0$, then $\deg a_{ij} = \deg g_j - \deg f_i$. Since $\det \phi = \sum_{\sigma \in S_r} \text{sgn}(\sigma) a_{1\sigma(1)} \cdots a_{r\sigma(r)}$ (where, S_r is a permutation group of r elements, and $\text{sgn}(\sigma)$ is the sign of a permutation σ), it suffices to show that $\deg a_{1\sigma(1)} \cdots a_{r\sigma(r)}$ is independent of σ , whenever $a_{i\sigma(i)} \neq 0$ for all $1 \leq i \leq r$. This is true, since if $a_{i\sigma(i)} \neq 0$ for all $1 \leq i \leq r$, then $\deg a_{1\sigma(1)} \cdots a_{r\sigma(r)} = \sum_{i=1}^r (\deg g_{\sigma(i)} - \deg f_i) = \sum_{i=1}^r (\deg g_i - \deg f_i)$, which is independent of σ . Radicals of homogeneous ideals are homogeneous. Minimal prime ideals of M are homogeneous. Therefore the ideals $J_{CM}(M)$ and $J_{(S_i)}(M)$ are homogeneous. Minimal prime ideals of homogeneous ideals are homogeneous, so the Cohen-Macaulay and (S_i) -loci of M are determined by homogeneous prime ideals. Hence to determine whether M has property (S_i) , (or, is Cohen-Macaulay), it suffices to check this at homogeneous prime ideals. We remark here that the above argument carries over *mutatis mutandis* to the situation of multigrading, for instance, when $M = R/I$ for a monomial ideal I .

3. PROPERTY $(N_{c,i})$ FOR ALEXANDER DUALS

To every square-free monomial ideal J in R , we can associate a simplicial complex Δ , called the *Stanley-Reisner complex* of J . See [MS05, Chapter 1]. For any monomial ideal J , R/J inherits the multigrading of R . For any multigraded R -module M , we define multigraded Betti numbers $\beta_{l,\sigma}(M) := \dim_{\mathbb{k}} \text{Tor}_l^R(\mathbb{k}, M)_\sigma$, where $1 \leq l \leq n$ and $\sigma \subseteq \mathbb{N}^n$ is a *multidegree*. When σ is square-free, *i.e.*, when the every entry in σ is 0 or 1, then we identify σ with the subset $\{x_i : \sigma_i \neq 0\}$, and, by abuse of notation, say that $\sigma \subseteq \{x_1, \dots, x_n\}$. For a simplicial complex Δ and square-free multidegree $\sigma \subseteq \{x_1, \dots, x_n\}$, we define $\Delta|_\sigma := \{F \in \Delta : F \subseteq \sigma\}$.

Proposition 12 (Hochster, [MS05, Corollary 5.12]). *Let J be a square-free monomial ideal and Δ its Stanley-Reisner complex. Non-zero multigraded Betti numbers of R/I occur at square-free multidegrees. Moreover, for a square-free multidegree $\sigma \subseteq \{x_1, \dots, x_n\}$,*

$$\beta_{i,\sigma}(J) = \beta_{i-1,\sigma}(R/J) = \dim_{\mathbb{k}} \tilde{H}_{|\sigma|-i-2}(\Delta|_\sigma; \mathbb{k}).$$

An immediate corollary to Hochster's formula is that $\text{depth } R/J = 1$ if and only if Δ is not connected: indeed, the Auslander-Buchsbaum formula implies that $\text{depth } R/J = 1$ if and only if $\text{Tor}_{n-1}^R(\mathbb{k}, R/J) \neq 0$. Since $\text{Tor}_i^R(\mathbb{k}, R/J)_\sigma = 0$ if $|\sigma| \leq i$, Hochster's formula gives the equivalence with $\text{Tor}_{n-1}^R(\mathbb{k}, R/J)_{\{x_1, \dots, x_n\}} \neq 0$, and, again, with $\tilde{H}_0(\Delta; \mathbb{k}) \neq 0$, which is equivalent to Δ being disconnected.

Proposition 13 (Terai [Ter99]; [MS05, Theorem 5.59]). *For any square-free monomial ideal J , $\text{pd } R/J = \text{reg } J^*$.*

Lemma 14. *With notation as above,*

- (a) *For all $1 \leq l \leq n$, $(I : x_l)^* = (I^* \cap \mathbb{k}[x_1, \dots, \hat{x}_l, \dots, x_n])R$.*
- (b) *If R/I satisfies (S_i) , then, for all $1 \leq l \leq n$, $R/(I : x_l)$ satisfies (S_i) .*

Proof. (a): Associated primes of $(I : x_l)$ are exactly those of I not containing x_l . Hence while computing the dual, we take the generators not involving x_l .

(b): It suffices to show that $J_{(S_i)}(R/(I : x_l)) = R$. By way of contradiction, if $J_{(S_i)}(R/(I : x_l)) \neq R$, then let \mathfrak{p} be a minimal prime ideal over $J_{(S_i)}(R/(I : x_l))$;

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hence $(R/(I : x_l))_{\mathfrak{p}}$ does not have property (S_i) . Since no monomial minimal generator of $(I : x_l)$ is divisible by x_l , \mathfrak{p} is a monomial ideal not containing x_l ; see Discussion 11. Therefore $(R/(I : x_l))_{\mathfrak{p}} \simeq (R/I)_{\mathfrak{p}}$, which has property (S_i) , a contradiction. \square

We are now ready to prove Theorem 1.

Theorem 1. *Then for $i > 1$, the following are equivalent:*

- (a) R/I satisfies property (S_i) .
- (b) The Alexander dual I^* satisfies $(N_{c,i})$.

Proof. We prove both the directions by induction on n . Let $n = 3$. For any non-zero ideal $I \subseteq R = \mathbb{k}[x_1, x_2, x_3]$, if R/I satisfies (S_2) (equivalently, since $\dim R/I \leq 2$, (S_i) for all $i \geq 2$), then R/I is Cohen-Macaulay, and, hence $\text{pd } R/I = \text{ht } I$. By Proposition 13, we see that $\text{reg } I^* = \text{ht } I$; however, since I^* is generated by monomials of degree $\text{ht } I$, I^* has a linear resolution; in particular, I^* has property $(N_{c,2})$. Conversely, if I^* has property $(N_{c,2})$, and $c = 1$, then R/I is a complete intersection, and Cohen-Macaulay. If $c = 2$, then $\dim R/I = 1$. One-dimensional reduced Noetherian local rings are Cohen-Macaulay.

(a) \implies (b): By way of contradiction, assume that I^* does not have the property $(N_{c,i})$. By induction, assume that n is the least integer for which there is such a counter-example. By Lemma 14(a), $(I : x_l)^*$ satisfies $(N_{c,i})$ for all $1 \leq l \leq n$. Now, since I does not have $(N_{c,i})$, there is a (square-free) multidegree σ and $j \leq i - 1$ such that $|\sigma| > j + c$ and $\beta_{j,\sigma}(I^*) \neq 0$. We now claim that $\sigma = \{x_1, \dots, x_n\}$: for, if, say, $x_1 \notin \sigma$, then let Δ be the Stanley-Reisner complex of I^* , and $\tilde{\Delta}$ of $(I^* \cap \mathbb{k}[x_2, \dots, x_n])R$. Then, by applying Hochster's formula, we have

$$\begin{aligned} \beta_{j,\sigma}(I^*) &= \dim_{\mathbb{k}} \tilde{H}_{|\sigma|-j-2}(\Delta|_{\sigma}; \mathbb{k}) = \dim_{\mathbb{k}} \tilde{H}_{|\sigma|-j-2}(\tilde{\Delta}|_{\sigma}; \mathbb{k}) \\ &= \beta_{j,\sigma}((I^* \cap \mathbb{k}[x_2, \dots, x_n])R) \\ &= \beta_{j,\sigma}((I : x_1)^*) \end{aligned}$$

contradicting the fact that $(I : x_1)^*$ satisfies $(N_{c,i})$. Hence $\sigma = \{x_1, \dots, x_n\}$, and, therefore, $j < n - c = \dim R/I$. By choice, $j < i$. Moreover, $\text{reg } I^* \geq n - j$. By Proposition 13, $\text{pd } R/I \geq n - j$, and, therefore $\text{depth } R/I \leq j$, contradicting the hypothesis that R/I satisfies (S_i) .

(b) \implies (a) : By way of contradiction, assume that R/I does not satisfy (S_i) . We may again assume that n is the least number of variables where such a counter-example exists. Since I^* satisfies $(N_{c,i})$, $(I : x_l)^*$ has $(N_{c,i})$ for all $1 \leq l \leq n$. By choice of n , $R/(I : x_l)$ satisfies (S_i) for all $1 \leq l \leq n$.

Now let $\mathfrak{p} \in \text{Spec } R$ be such that $\text{depth}(R/I)_{\mathfrak{p}} < \min\{i, \dim(R/I)_{\mathfrak{p}}\}$. If $x_l \notin \mathfrak{p}$, then, $(R/I)_{\mathfrak{p}} \simeq (R/(I : x_l))_{\mathfrak{p}}$. Hence $\text{depth}(R/I)_{\mathfrak{p}} \geq \min\{i, \dim(R/I)_{\mathfrak{p}}\}$. Therefore $\mathfrak{p} = \mathfrak{m}$. Hence $\text{depth } R/I < \min\{i, \dim R/I\}$. By Auslander-Buchsbaum formula, $\text{pd } R/I > n - i$. Again, by the result of Terai, $\text{reg } I^* > n - i$, *i.e.*, there exists j and a multidegree σ such that $\beta_{j,\sigma}(I^*) \neq 0$ and $|\sigma| - j > n - i$. By Hochster's theorem, non-zero Betti numbers are in square-free multidegrees, so, $|\sigma| \leq n$. Hence $j < i$, contradicting the hypothesis that I^* has $(N_{c,i})$. \square

Before we proceed, we observe that if $\dim R/I \geq 2$ and R/I is connected in codimension 1, then Stanley-Reisner complex Δ of I is connected; in fact, it is *strongly connected*, *i.e.*, for any two faces F and F' of Δ of maximal dimension, we can find a sequence $F_0 = F, F_1, \dots, F_r = F'$ of faces of maximal dimension such

that for all $1 \leq i \leq n-1$, $F_i \cap F_{i-1}$ is a face of codimension 1 in F_i and F_{i-1} . Using the correspondence between faces of Δ and prime ideals containing I [MS05, Theorem 1.7], it suffices to show that for any $\mathfrak{p}, \mathfrak{p}' \in \text{Ass } R/I$, there is a sequence $\mathfrak{p}_0 = \mathfrak{p}, \mathfrak{p}_1, \dots, \mathfrak{p}_r = \mathfrak{p}'$ of associated primes of R/I such that for all $1 \leq i \leq n-1$, $\text{ht}(\mathfrak{p}_i + \mathfrak{p}_{i+1}) = \text{ht } \mathfrak{p}_i + 1 = \text{ht } \mathfrak{p}_{i+1} + 1$. This follows from setting $d = 2$ in [EGA, IV, 5.10.8]. Finally, since R/I is connected in codimension 1, it is equidimensional; this is the content of the proof of [EGA, IV, 5.10.9]. Hence every vertex of Δ is in some face of maximal dimension, so Δ is connected.

Theorem 6. *Let $R = \mathbb{k}[x_1, \dots, x_n]$ be a polynomial ring in n variables and let $I \subseteq R$ be a square-free monomial ideal. Then $\text{Spec } R/I$ is locally connected in codimension 1 if and only if R/I satisfies property (S_2) .*

Proof. We will show that if $\text{Spec } R/I$ is locally connected in codimension 1, then R/I has property (S_i) ; the other implication is already known [Har62, Corollary 2.4]. If $c \geq n-1$, then it is clear that R/I is locally connected in codimension 1 and that R/I has property (S_2) . Therefore we will assume that $c \leq n-2$.

We proceed by induction on n . Let $n = 3$. It is easy to verify that any unmixed monomial ideal in three variables is locally connected in codimension 1. Since $c = 1$, R/I is a complete intersection and, hence has property (S_2) . Now assume that $n > 3$.

We first observe that for all $1 \leq l \leq n$, $\text{Spec } R/(I : x_l)$ is locally connected in codimension 1, because, as topological spaces, $\text{Spec } R/(I : x_l)$ is homeomorphic to $\text{Spec}(R/I)_{x_l}$, which is locally connected in codimension 1, $(R/I)_{x_l}$ being a localization of R/I . Since x_l does not divide any minimal generator of $(I : x_l)$, $(I : x_l)$ is extended from the subring $\mathbb{k}[x_1, \dots, \widehat{x}_l, \dots, x_n] \subseteq R$. By induction $R/(I : x_l)$ has property (S_2) . Now let $\mathfrak{p} \in \text{Spec } R, \mathfrak{p} \neq \mathfrak{m}$. We can then pick $x_l \notin \mathfrak{p}$. Since $(R/I)_{\mathfrak{p}} \simeq (R/(I : x_l))_{\mathfrak{p}}$, we see that $\text{depth}(R/I)_{\mathfrak{p}} \geq \min\{2, \dim(R/I)_{\mathfrak{p}}\}$. It remains to show that $\text{depth } R/I \geq 2$, *i.e.*, that the Stanley-Reisner complex Δ of I is connected, which follows from the preceding discussion. \square

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