

## Exam 2 - Practice problems (Ch 7 only)

1. Find the solution of the following initial value problem:

$$\vec{x}' = \begin{pmatrix} -1 & 2 & -2 \\ 0 & 1 & 0 \\ -2 & 2 & -1 \end{pmatrix} \vec{x}, \quad \vec{x}(0) = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix}$$

*Solution.* First we find the general solution and then the particular one that satisfies the initial condition. Start by solving  $\det(A - \lambda I) = 0$  to find the eigenvalues:

$$\begin{vmatrix} -1 - \lambda & 2 & -2 \\ 0 & 1 - \lambda & 0 \\ -2 & 2 & -1 - \lambda \end{vmatrix} = (1 - \lambda)(\lambda^2 + 2\lambda - 3) = 0 \quad \implies \quad \lambda_1 = \lambda_2 = 1, \lambda_3 = -3.$$

For the eigenvalues  $\lambda_1 = \lambda_2 = 1$  we expect to find one or two linear independent eigenvectors by solving  $(A - I) \cdot \vec{\xi} = \mathbf{0}$ :

$$\begin{pmatrix} -2 & 2 & -2 \\ 0 & 0 & 0 \\ -2 & 2 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \implies \quad -\xi_1 + \xi_2 - \xi_3 = 0$$

and hence we can choose 2 linear independent eigenvectors

$$\vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \implies \quad \vec{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^t, \quad \vec{x}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^t.$$

For the eigenvalue  $\lambda_3 = -3$  we expect to find one eigenvector (I really mean one eigenvector and all its multiples or, in other words, one linear independent eigenvector):

$$\begin{pmatrix} 2 & 2 & -2 \\ 0 & 4 & 0 \\ -2 & 2 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \implies \quad \xi_2 = 0, \quad \xi_1 - \xi_3 = 0.$$

Choose

$$\vec{\xi}^{(3)} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \implies \quad \vec{x}^{(3)} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{-3t}.$$

The *general solution* is then

$$\vec{x} = C_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^t + C_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^t + C_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{-3t} = \begin{pmatrix} e^t & 0 & e^{-3t} \\ e^t & e^t & 0 \\ 0 & e^t & e^{-3t} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix},$$

the matrix above being the fundamental matrix  $\Psi(t)$ . Finding the solution of the initial value problem requires finding the  $C$ s for which:

$$\vec{x}(0) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} \quad \implies \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} \quad \implies \quad \begin{matrix} C_1 = 2 \\ C_2 = 1 \\ C_3 = 1 \end{matrix}$$

(I have subtracted the first line from the second and the resulting second line from the third, also doing the same operations on the vector on the right side).

The solution is

$$\vec{x} = 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{-3t}.$$

2. Find the general real solution for

$$\vec{x}' = \begin{pmatrix} -1 & 1 \\ -4 & -1 \end{pmatrix} \vec{x}$$

and also a solution passing through the point  $\vec{x}_0 = (1, -2)$ . Sketch the phase portrait and decide the type and stability of the equilibrium solution  $\vec{x} = 0$ .

*Solution.*  $\det(A - \lambda I) = \lambda^2 + 2\lambda + 5$  has complex conjugate solutions  $-1 \pm 2i$ . To find the eigenvector associated to  $-1 + 2i$  solve:

$$\begin{pmatrix} -2i & 1 \\ -4 & -2i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies -2i\xi_1 + \xi_2 = 0. \text{ Take } \vec{\xi} = \begin{pmatrix} 1 \\ 2i \end{pmatrix}.$$

This gives one solution  $\vec{x}^{(1)}$ ; another solution would be the conjugate of this one. However, we want two real linear independent solutions; these can be obtained by collecting the real and imaginary part of  $\vec{x}^{(1)}$ :

$$\vec{x}^{(1)} = \begin{pmatrix} 1 \\ 2i \end{pmatrix} e^{(-1+2i)t} = \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right] e^{-t} (\cos(2t) + i \sin(2t)) = e^{-t} \begin{pmatrix} \cos 2t \\ -2 \sin 2t \end{pmatrix} + i e^{-t} \begin{pmatrix} \sin 2t \\ 2 \cos 2t \end{pmatrix}.$$

The *general solution* is then

$$\vec{x} = c_1 e^{-t} \begin{pmatrix} \cos 2t \\ -2 \sin 2t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} \sin 2t \\ 2 \cos 2t \end{pmatrix}.$$

One solution that passes through  $\vec{x}_0 = (1, -2)$  can be obtained by asking to pass through  $\vec{x}_0$  at  $t = 0$ :

$$\vec{x}(0) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \implies \begin{matrix} c_1 = 1 \\ c_2 = -1 \end{matrix} \implies \vec{x} = e^{-t} \begin{pmatrix} \cos 2t - \sin 2t \\ -2 \sin 2t - 2 \cos 2t \end{pmatrix}.$$

Because the eigenvalues are complex,  $x = 0$  is a spiral point and, since their real part ( $= -1$ ) is negative, it is asymptotically stable.

3. Find the general solution of

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} -t \\ 1 \end{pmatrix}.$$

*Solution.* We solve first the associated homogeneous system. The eigenvalues are  $\lambda_1 = \lambda_2 = 0$ . It turns out that the matrix does not have two linear independent eigenvectors:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \xi_2 = 0. \text{ Take } \vec{\xi} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies \vec{x}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

A second solution would be given by  $\vec{x}^{(2)} = \vec{\xi} t e^{0 \cdot t} + \vec{\eta} e^{0 \cdot t}$  where  $\vec{\eta}$  is the solution of  $(A - \lambda I)\vec{\eta} = \vec{\xi}$ :

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies \vec{\eta} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \implies \vec{x}^{(2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} t \\ 1 \end{pmatrix}$$

The fundamental matrix is

$$\Psi(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Returning to the nonhomogeneous system, its general solution is given by  $\vec{x} = \Psi(t)\vec{u}$  where  $\vec{u}$  satisfies

$$\Psi(t)\vec{u}' = \begin{pmatrix} -t \\ 1 \end{pmatrix} \implies \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} -t \\ 1 \end{pmatrix} \implies \begin{matrix} u_1' = -2t \\ u_2' = 1 \end{matrix} \implies \begin{matrix} u_1 = -t^2 + c_1 \\ u_2 = t + c_2 \end{matrix}$$

The general solution is then:

$$\vec{x} = (-t^2 + c_1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (t + c_2) \begin{pmatrix} t \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} t \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ t \end{pmatrix}$$