# Linear Algebra

## Lesson 7

### Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix. A number(real or complex) $\lambda$ is an *eigenvalue* for A if

$$det(A - \lambda I) = 0$$

If we compute out $det(A - \lambda I)$, we get a polynomial

$$P(\lambda) = det(A - \lambda I)$$

called the *characteristic polynomial.*

Since $P(\lambda)$ is a polynomial of degree n and the eigenvalues are the roots of $u$, there are at most n eigenvalues for an $n \times n$ matrix.

If $\lambda$ is an eigenvalue, then since $det(A - \lambda I) = 0$, $A - \lambda I$ is singular and thus

$$(A - \lambda I)x = 0 \quad or \quad Ax = \lambda x$$

has nontrivial solutions. Such a nontrivial solution is called an *eigenvector* for $\lambda$. The solution space of $(A - \lambda I)x = 0$ is called the *eigenspace* for $\lambda$. (Thus the eigenspace for $\lambda$ is the set of all eigenvectors for $\lambda$ with the 0 vector also thrown in.)

Notice that $\lambda = 0$ is a perfectly good candidate, and will occur in the list of eigenvalues when A itself is singular.

The multiplicity of the root $\lambda$ of $P(\lambda)$ is called the *algebraic multiplicity*.

The *geometric multiplicity* is the dimension of the eigenspace.

We can see from the analysis that we have for any $n \times n$ matrix,

$$geometric\ mult. \leq algebraic\ mult. \leq n$$

Some odd fact:

1. If A $n \times n$ is upper(lower) triangular, then the eigenvalues are displayed on the diagonal.

2. Row operations on A *change* its eigenvectors.

# Lesson8

## Orthogonal and Unitary matrices

We continue to consider only square matrix. A is an $n \times n$ matrix with *real* entries.
Theorem 1: If A is symmetric, its eigenvalues are pure real. If A is skew-symmetric, eigenvalues are pure imaginary or 0.

A *real* matrix B is an *orthogonal matrix* if $B^T = B^{-1}$.

Theorem: An $n \times n$ matrix B is orthogonal if and only if its rows form an orthonormal set. The same is true for its columns.

Theorem 2: If B is an orthogonal matrix, then its eigenvalues $\lambda$ (which may be complex) satisfy $|\lambda| = 1$.

Orthogonal matrices and symmetric matrices have real entries. When it is necessary to deal with complex matrices,we need to extend these notion.

First, we extend the concept of inner product to n component vectors with complex entries. The set of such vectors is denoted by $C^n$.

$$(u, v) = \bar{u}^T v, u, v \in C^n$$

Now,we extend the notion of symmetric (real matrices) to *Hermitian* matrices(complex matrices). A is Hermitian if $\bar{A}^T = A$ or (or $A^T = \bar{A}$). (*skew-Hermitian* if $\bar{A}^T = -A$)

Orthogonal matrices become *unitary* matrices in the complex case. B is unitary if

$$\bar{B}^T = B^{-1}$$

Note: Eigenvalues of complex matrices need not occur in conjugate pairs since the characteristic polynomial has complex coefficients.

Theorem $1'$: The eigenvalues of a Hermitian matrix are real. The eigenvalues of a skew-Hermitian matrix are imaginary or 0.

Theorem $2'$: If $\lambda$ is an eigenvalue of a unitary matrix B, then $|\lambda| = 1$.

An important property of unitary (and hence orthogonal) matrices is that they *preserve inner product.*

A *quadratic form* in $R^n$ is an expression

$$x^T A x \quad x \in R^n$$

In the general case, we get

$$x^T A x = \sum_{i,j=1}^{N} a_{ij} x_i x_j$$

A quadratic form is *positive definite* if $x^T A x > 0$ for all $x$ except $x = 0$.