7.2: Review of Matrices

- For theoretical and computational reasons, we review results of matrix theory in this section and the next.

- A **matrix A** is an *m* x *n* rectangular array of elements, arranged in *m* rows and *n* columns, denoted

$$\mathbf{A} = \left(a_{ij}\right) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

- Some examples of 2 x 2 matrices are given below:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 3-2i \\ 4+5i & 6-7i \end{pmatrix}$$

Transpose

- The **transpose** of $\mathbf{A} = (a_{ij})$ is $\mathbf{A}^T = (a_{ji})$.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \Rightarrow \mathbf{A}^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

- For example,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \Rightarrow \mathbf{A}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \Rightarrow \mathbf{B}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Conjugate

- The **conjugate** of $\mathbf{A} = (a_{ij})$ is $\overline{\mathbf{A}} = (\overline{a_{ij}})$.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \Rightarrow \quad \overline{\mathbf{A}} = \begin{pmatrix} \overline{a}_{11} & \overline{a}_{12} & \cdots & \overline{a}_{1n} \\ \overline{a}_{21} & \overline{a}_{22} & \cdots & \overline{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a}_{m1} & \overline{a}_{m2} & \cdots & \overline{a}_{mn} \end{pmatrix}$$

- For example,

$$\mathbf{A} = \begin{pmatrix} 1 & 2+3i \\ 3-4i & 4 \end{pmatrix} \Rightarrow \overline{\mathbf{A}} = \begin{pmatrix} 1 & 2-3i \\ 3+4i & 4 \end{pmatrix}$$

Adjoint

- The **adjoint** of $\mathbf{A}$ is $\overline{\mathbf{A}}^T$, and is denoted by $\mathbf{A}^*$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \Rightarrow \mathbf{A}^* = \begin{pmatrix} \bar{a}_{11} & \bar{a}_{21} & \cdots & \bar{a}_{m1} \\ \bar{a}_{12} & \bar{a}_{22} & \cdots & \bar{a}_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{1n} & \bar{a}_{2n} & \cdots & \bar{a}_{mn} \end{pmatrix}$$

- For example,

$$\mathbf{A} = \begin{pmatrix} 1 & 2+3i \\ 3-4i & 4 \end{pmatrix} \Rightarrow \mathbf{A}^* = \begin{pmatrix} 1 & 3+4i \\ 2-3i & 4 \end{pmatrix}$$

Square Matrices

- A **square matrix A** has the same number of rows and columns. That is, **A** is n x n. In this case, **A** is said to have order n.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

- For example,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Vectors

- A **column vector x** is an *n* x 1 matrix. For example,

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

- A **row vector x** is a 1 x *n* matrix.   For example,

$$\mathbf{y} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

- Note here that $\mathbf{y} = \mathbf{x}^T$, and that in general, if **x** is a column vector **x**, then $\mathbf{x}^T$ is a row vector.

The Zero Matrix

- The **zero matrix** is defined to be $\mathbf{0} = (0)$, whose dimensions depend on the context. For example,

$$\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \ldots$$

Matrix Equality

- Two matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ are **equal** if $a_{ij} = b_{ij}$ for all i and j. For example,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \Rightarrow \mathbf{A} = \mathbf{B}$$

(Question)

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{A} = \mathbf{B}\,?$$

Matrix – Scalar Multiplication

- The product of a matrix $\mathbf{A} = (a_{ij})$ and a constant k is defined to be $\mathrm{k}\mathbf{A} = (\mathrm{k}a_{ij})$. For example,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \Rightarrow -5\mathbf{A} = \begin{pmatrix} -5 & -10 & -15 \\ -20 & -25 & -30 \end{pmatrix}$$

Matrix Addition and Subtraction

- The **sum** of two m x n matrices **A** = (a_{ij}) and **B** = (b_{ij}) is defined to be **A** + **B** = (a_{ij} + b_{ij}). For example,

$$\mathbf{A}=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix},\ \mathbf{B}=\begin{pmatrix}5 & 6\\ 7 & 8\end{pmatrix}\ \Rightarrow \mathbf{A}+\mathbf{B}=\begin{pmatrix}6 & 8\\ 10 & 12\end{pmatrix}$$

- The **difference** of two m x n matrices **A** = (a_{ij}) and **B** = (b_{ij}) is defined to be **A** - **B** = (a_{ij} - b_{ij}). For example,

$$\mathbf{A}=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix},\ \mathbf{B}=\begin{pmatrix}5 & 6\\ 7 & 8\end{pmatrix}\ \Rightarrow \mathbf{A}-\mathbf{B}=\begin{pmatrix}-4 & -4\\ -4 & -4\end{pmatrix}$$

Matrix Multiplication

- The **product** of an *m* x *n* matrix **A** = (a_{ij}) and an *n* x *r* matrix **B** = (b_{ij}) is defined to be the matrix **C** = (c_{ij}), where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

- Examples (note **AB** does not necessarily equal **BA**):

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \Rightarrow \mathbf{AB} = \begin{pmatrix} 1+2*2 & 3+2*4 \\ 3+4*2 & 3*3+4*4 \end{pmatrix} = \begin{pmatrix} 5 & 11 \\ 11 & 25 \end{pmatrix}$$

$$\Rightarrow \mathbf{BA} = \begin{pmatrix} 1+9 & 2+12 \\ 2+12 & 4+16 \end{pmatrix} = \begin{pmatrix} 10 & 14 \\ 14 & 20 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 3 & 0 \\ 1 & 2 \\ 0 & -1 \end{pmatrix} \Rightarrow \mathbf{CD} = \begin{pmatrix} 3+2+0 & 0+4-3 \\ 12+5+0 & 0+10-6 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 17 & 4 \end{pmatrix}$$

Example 1: Matrix Multiplication

- To illustrate matrix multiplication and show that it is not commutative, consider the following matrices:

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -1 \\ 2 & 1 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 0 \\ 2 & -1 & 1 \end{pmatrix}$$

- From the definition of matrix multiplication we have:

$$\mathbf{AB} = \begin{pmatrix} 2-2+2 & 1+2-1 & -1+1 \\ 2-2 & -2+1 & -1 \\ 4+1+2 & 2-1-1 & -2+1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 0 & -1 & -1 \\ 7 & 0 & -1 \end{pmatrix}$$

$$\mathbf{BA} = \begin{pmatrix} 2-2 & -4+2-1 & 2-1-1 \\ 1 & -2-2 & 1+1 \\ 2+2 & -4-2+1 & 2+1+1 \end{pmatrix} = \begin{pmatrix} 0 & -3 & 0 \\ 1 & -4 & 2 \\ 4 & -5 & 4 \end{pmatrix} \neq \mathbf{AB}$$

Vector Multiplication

- The **dot product** of two n x 1 vectors **x** & **y** is defined as

$$\mathbf{x}^T\mathbf{y} = \sum_{k=1}^{n} x_i y_j$$

- The **inner product** of two n x 1 vectors **x** & **y** is defined as

$$(\mathbf{x},\mathbf{y}) = \mathbf{x}^T\overline{\mathbf{y}} = \sum_{k=1}^{n} x_i \overline{y}_j \qquad (\mathbf{x},\mathbf{y}) = \mathbf{x}^T A\overline{\mathbf{y}} = \sum_{k=1}^{n} a_{ij} x_i \overline{y}_j$$

- Example:

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3i \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -1 \\ 2-3i \\ 5+5i \end{pmatrix} \Rightarrow \mathbf{x}^T\mathbf{y} = (1)(-1)+(2)(2-3i)+(3i)(5+5i) = -12+9i$$

$$\Rightarrow (\mathbf{x},\mathbf{y}) = \mathbf{x}^T\overline{\mathbf{y}} = (1)(-1)+(2)(2+3i)+(3i)(5-5i) = 18+21i$$

Vector Length

- The **length** of an n x 1 vector $\mathbf{x}$ is defined as

$$\|\mathbf{x}\| = (\mathbf{x},\mathbf{x})^{1/2} = \left[\sum_{k=1}^{n} x_k \bar{x}_k\right]^{1/2} = \left[\sum_{k=1}^{n} | x_k |^2\right]^{1/2}$$

- Note here that we have used the fact that if $x = a + bi$, then

$$x \cdot \bar{x} = (a+bi)(a-bi) = a^2 + b^2 = |x|^2$$

- Example:

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3+4i \end{pmatrix} \quad \Rightarrow \quad \|\mathbf{x}\| = (\mathbf{x},\mathbf{x})^{1/2} = \sqrt{(1)(1) + (2)(2) + (3+4i)(3-4i)}$$

$$= \sqrt{1 + 4 + (9+16)} = \sqrt{30}$$

Orthogonality

- Two n x 1 vectors **x** & **y** are **orthogonal** if $(\mathbf{x},\mathbf{y}) = \mathbf{x}\cdot\mathbf{y} = 0$.
  i.e. the angle is 90°

- Example:

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 11 \\ -4 \\ -1 \end{pmatrix} \Rightarrow (\mathbf{x},\mathbf{y}) = (1)(11) + (2)(-4) + (3)(-1) = 0$$

Identity Matrix

- The multiplicative **identity matrix I** is an *n* x *n* matrix given by

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

- For any square matrix **A**, it follows that **AI** = **IA** = **A**.

- The dimensions of **I** depend on the context. For example,

$$\mathbf{AI} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad \mathbf{IB} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Inverse Matrix

- A square matrix **A** is **nonsingular**, or **invertible**, if there exists a matrix **B** such that that **AB** = **BA** = **I**. Otherwise **A** is **singular**.

- The matrix **B**, if it exists, is unique and is denoted by $\mathbf{A}^{-1}$ and is called the **inverse** of **A**.

- It turns out that $\mathbf{A}^{-1}$ exists iff $\det\mathbf{A} \neq 0$, and $\mathbf{A}^{-1}$ can be found using **row reduction** (also called Gaussian elimination) on the augmented matrix **(A|I)**, see example on next slide.

- The three elementary row operations: detA is not changed.
  - Interchange two rows.
  - Multiply a row by a nonzero scalar.
  - Add a multiple of one row to another row.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Example 2: Finding the Inverse of a Matrix (1 of 2)

- Use Gaussian elimination or elementary row operation to find the inverse of the matrix **A** below, if it exists.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad (\mathbf{A}|\mathbf{I}) = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{pmatrix}, \qquad (\mathbf{A}|\mathbf{I}) = \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}.$$

- Solution: If possible, use elementary row operations to reduce **(A|I)**,

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & -1 \\ 3 & -1 & 2 \\ 2 & 2 & 3 \end{pmatrix} \qquad (\mathbf{A}|\mathbf{I}) = \begin{pmatrix} 1 & -1 & -1 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{pmatrix},$$

such that the left side is the identity matrix, for then the right side will be $\mathbf{A}^{-1}$. (See next slide.)

Example 2: Finding the Inverse of a Matrix (2 of 2)

$$(\mathbf{A}|\mathbf{I}) = \begin{pmatrix} 1 & -1 & -1 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 5 & -3 & 1 & 0 \\ 0 & 4 & 5 & -2 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 5/2 & -3/2 & 1/2 & 0 \\ 0 & 4 & 5 & -2 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3/2 & -1/2 & 1/2 & 0 \\ 0 & 1 & 5/2 & -3/2 & 1/2 & 0 \\ 0 & 0 & -5 & 4 & -2 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 3/2 & -1/2 & 1/2 & 0 \\ 0 & 1 & 5/2 & -3/2 & 1/2 & 0 \\ 0 & 0 & -5 & 4 & -2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 7/10 & -1/10 & 3/10 \\ 0 & 1 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & -4/5 & 2/5 & -1/5 \end{pmatrix}$$

- Thus $$\mathbf{A}^{-1} = \begin{pmatrix} 7/10 & -1/10 & 3/10 \\ 1/2 & -1/2 & 1/2 \\ -4/5 & 2/5 & -1/5 \end{pmatrix}$$

Matrix Functions

- The elements of a matrix can be functions of a real variable. In this case, we write

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_m(t) \end{pmatrix}, \quad \mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}(t) & a_{m2}(t) & \cdots & a_{mn}(t) \end{pmatrix}$$

- Such a matrix is continuous at a point, or on an interval (a, b), if each element is continuous there. Similarly with differentiation and integration:

$$\frac{d\mathbf{A}}{dt} = \left(\frac{da_{ij}}{dt} \right), \quad \int_a^b \mathbf{A}(t)dt = \left(\int_a^b a_{ij}(t)dt \right)$$

Example & Differentiation Rules

- Example: $$\mathbf{A}(t) = \begin{pmatrix} 3t^2 & \sin t \\ \cos t & 4 \end{pmatrix} \Rightarrow \frac{d\mathbf{A}}{dt} = \begin{pmatrix} 6t & \cos t \\ -\sin t & 0 \end{pmatrix},$$ $$\Rightarrow \int_0^\pi \mathbf{A}(t)dt = \begin{pmatrix} \pi^3 & 0 \\ -1 & 4\pi \end{pmatrix}$$

- Many of the rules from calculus apply in this setting. For example:

$$\frac{d(\mathbf{CA})}{dt} = \mathbf{C}\frac{d\mathbf{A}}{dt}, \quad \text{where } \mathbf{C} \text{ is a constant matrix}$$

$$\frac{d(\mathbf{A}+\mathbf{B})}{dt} = \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{B}}{dt}$$

$$\frac{d(\mathbf{AB})}{dt} = \left(\frac{d\mathbf{A}}{dt}\right)\mathbf{B} + \mathbf{A}\left(\frac{d\mathbf{B}}{dt}\right)$$