# Homework #3

**Section 2.3**

**#17** Are there non-trivial solutions

\[
\begin{align*}
  x + 2y + 3z &= 0, \\
  2x + 3z &= 0, \\
  x + 2y + 3z &= 0.
\end{align*}
\]

\[
\begin{pmatrix}
  1 & 2 & 3 & 0 \\
  0 & 2 & 3 & 0 \\
  1 & 2 & 3 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & \frac{3}{2} & 0 \\
  0 & 0 & 0 & 0
\end{pmatrix}
\]

no leading one

(2 is a free variable)

\[
\begin{pmatrix}
  1 & 2 & 3 \\
  0 & 2 & 3 \\
  1 & 2 & 3
\end{pmatrix}
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix}
= 0
\]

infinitely many solutions \iff \det \begin{pmatrix}
  1 & 2 & 3 \\
  0 & 2 & 3 \\
  1 & 2 & 3
\end{pmatrix} \neq 0

\begin{pmatrix}
  1 & 2 & 3 \\
  0 & 2 & 3 \\
  1 & 2 & 3
\end{pmatrix} = 0

\]

**#19** What values of \( a \) is \( \begin{pmatrix}
  1 & 1 & 0 \\
  1 & 0 & a \\
  1 & 2 & a
\end{pmatrix} \) invertible?

\[
\begin{pmatrix}
  1 & 1 & 0 \\
  1 & 0 & 0 \\
  1 & 2 & a
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  \_ & \_ & \_ \\
  \_ & \_ & \_ \\
  \_ & \_ & \_ \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
  1 & 1 & 0 \\
  0 & 1 & 0 \\
  1 & 1 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  \_ & \_ & \_ \\
  \_ & \_ & \_ \\
  \_ & \_ & \_ \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
  1 & 1 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  \_ & \_ & \_ \\
  \_ & \_ & \_ \\
  \_ & \_ & \_ \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
  1 & 1 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{pmatrix}
\]
Is \( \det \left( \begin{array}{ccc} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & a \end{array} \right) \neq 0 \)?

\[
\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = -1 \begin{vmatrix} 1 & 0 \\ 1 & a \end{vmatrix} = -a \neq 0 \\
\text{when } a \neq 0.
\]
Chapter 4 - Vector Spaces

Section 4.1 - Vector spaces in plane, 3-dim space (and \( \mathbb{R}^n \))

Vectors in the plane

\[ \vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} \]

Physics - an object with magnitude (length) & direction.

**notation**: \( O = (-1, 1) \) & \( P = (2, 2) \)

then \( \overrightarrow{OP} \) is vector with tail \((-1, 1)\) & head \((2, 2)\)

\[ \overrightarrow{OP} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \]

Similar for vectors in \( \mathbb{R}^3 \)

\[ \vec{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \]

\[ \vec{u} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \]

Geometry of addition & scalar multi.
\[ \vec{u} + \vec{v} = \vec{v} + \vec{u} \]

\[ -2\vec{u} \]

\[ 3\vec{u} \]
4.2 - Abstract Vector Spaces

**Generalization:** Look at what we are really using here?

Idea here is that what's really important in \( \mathbb{R}^n \) is that we can

1. add vectors together
2. multiply vectors by real numbers (scalar multiplication).

**Def** A real vector space is a set \( V \) of elements with two operations \( + \) and \( \cdot \) with the following properties:

- **a)** If \( \vec{u}, \vec{v} \) are in \( V \), then \( \vec{u} + \vec{v} \) is in \( V \)
  1. \( \vec{u} + \vec{v} = \vec{v} + \vec{u} \) (commutative)
  2. \( \vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w} \) (associative)
  3. There is an element \( \vec{0} \) in \( V \) such that \( \vec{0} + \vec{u} = \vec{u} \) (zero vector exists)
  4. For every \( \vec{u} \) in \( V \), there is an element \( -\vec{u} \) in \( V \) such that \( \vec{u} + -\vec{u} = \vec{0} \) (negative exists).

- **b)** If \( \vec{u} \) is in \( V \) and \( c \) is a real number, then \( c \cdot \vec{u} \) is in \( V \)
  5. \( c \cdot (\vec{u} + \vec{v}) = (c \cdot \vec{u}) + (c \cdot \vec{v}) \) (distribution properties)
  6. \( (c + d) \cdot \vec{u} = (c \cdot \vec{u}) + (d \cdot \vec{u}) \)
  7. \( c \cdot (d \cdot \vec{u}) = (cd) \cdot \vec{u} \) (associative)
  8. \( 1 \cdot \vec{u} = \vec{u} \) (multiply by 1).

**Examples**

1. \( \mathbb{R}^n \) (Section 4.1)
2. \( M_{mn} = \) collection of all \( m \times n \) matrices.
II) \( M_{mn} = \) collection of all \( m \times n \) matrices.

"vector" are \( m \times n \) matrices.

\( \oplus \) is regular matrix addition \( A \oplus B = A + B \)
\( \odot \) is regular scalar multiplication \( c \odot A = cA \)

Zero element \( \mathbf{0} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \)

\( A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad -A = \begin{pmatrix} -1 & -2 \\ -3 & -4 \end{pmatrix} \)

III) \( P_n = \) all polynomials of degree \( \leq n \).

\( p(t) = a_n t^n + a_{n-1} t^{n-1} + \ldots + a_1 t + a_0 \) \( (a_0, a_1, \ldots, a_n \text{ are fixed constants}) \).

"vector" are polynomials.

\( \oplus \) is normal way to add polynomials
\( \odot \) is normal way to multiply polynomials by constant

\( q(t) = b_n t^n + b_{n-1} t^{n-1} + \ldots + b_1 t + b_0 \)

\( p \oplus q \) is polynomial \( (a_n + b_n) t^n + \ldots + (a_1 + b_1) t + (a_0 + b_0) \)

Question: What is \( \mathbf{0} \)?

- answer polynomial with all coefficients zero.

\( p(t) = 0 \).

IV) \( V = \) all real-valued continuous functions \( f : \mathbb{R} \rightarrow \mathbb{R} \)

( notation \( C(-\infty, \infty) \) for this space).
"vectors" are functions

"zero vector" is zero function \( f(t) = 0 \).

\( \square \) (Exercise 12 in homework)

\( V = (0, \infty) \) but with different rule for addition + scalar mult.

\[ u \oplus v = uv \] ("addition" is regular multiplication)

\[ c \odot u = u^c \] ("scalar mult." is exponatation).

Non-examples of Vector Spaces

1. \( V = \text{First quadrant of } \mathbb{R}^2 \)
   - closed under addition \( \checkmark \)
   - closed under scalar-mult. \( \times \)
   - Not a vector space.

2. \( V = \mathbb{R}^n \) but with a different addition rule.
   \[ \vec{v} \oplus \vec{u} = \vec{v} - \vec{u} \]
   \[ c \odot \vec{v} = c \vec{v} \] (normal)

   - closed under addition \( \checkmark \)
   - scalar mult. \( \checkmark \)
   - 8 more properties to check.

   1. addition commutative \( \vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u} ? \)
      \[ \vec{u} - \vec{v} = \vec{v} - \vec{u} \]
\[ \vec{u} - \vec{v} = \vec{v} - \vec{u} \]

Not a vector space.
Subspaces (Section 4.3)

Some subsets of vector spaces are also vector spaces.

Example \( V = \{(x, y, z) : z = x + y\}\)

**Definition**

If \( V \) is a vector space and \( W \subseteq V \) is also a vector space (with the same rules for \( + \) and \( 0 \)), then \( W \) is called a **subspace**.

**Theorem 4.3**

If \( V \) is a vector space, \( W \subseteq V \) is also a vector space (a subspace) if and only if:

1. \( W \) is closed under addition.
2. \( W \) is closed under scalar multiplication.

For subspaces, remaining 8 properties hold since we know they are true for \( V \):

\[
\alpha(u + v) = (\alpha u) + (\alpha v)
\]

**Examples (to non-examples)**

1. Any plane through the origin in \( \mathbb{R}^3 \) is a subspace.
2. First quadrant of $\mathbb{R}^2$ is not a subspace.
   (not closed under scalar mult.)

3. $V \subseteq M_{22} \quad V = \{ (a, b) : a + d = 0 \} \quad \text{trace} = 0.$

*Remark* For any square matrix $A$, the trace of $A$, $\text{tr}(A)$
   is the sum of its diagonal entries.

- **Closed under addition.** (YES)
  
  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ with $a + d = 0$, $x + w = 0.$
  
  $A + B = \begin{pmatrix} a + x & b + y \\ c + z & d + w \end{pmatrix} \quad a + x + d + w = 0$  

- **Closed under scalar mult.** (YES).
  
  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a + d = 0$  

  $rA = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix} \quad ra + rd = r(a + d) = 0.$

4. $U \subseteq M_{22}$, $U =$ invertible $2 \times 2$ matrices.

   Not a subspace  

   $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

   $A + B = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ \quad $\det(A + B) = 0$  

   $A + B$ is not invertible

   $U$ is not closed under addition.
**Linear Combinations + Subspaces**

**Def** A linear combination of vectors \( \vec{v}_1, \ldots, \vec{v}_k \) is a vector of the form

\[
a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots + a_k \vec{v}_k,
\]

where \( a_i, a_2, \ldots, a_k \in \mathbb{R} \).

**Example** \( W \subseteq \mathbb{R}^3 \) is the collection of all linear combinations of \( \vec{v}_1 = (1, 0, 0) \) and \( \vec{v}_2 = (0, 1, 0) \).

Elements of \( W \):

\[
\begin{align*}
(1, 0, 0) &= 1 \vec{v}_1 + 0 \vec{v}_2 \\
(1, 1, 0) &= \vec{v}_1 + \vec{v}_2 \\
(0, 2, 0) &= 2 \vec{v}_1 - \vec{v}_2 \\
(-1, 1, 0) &= -\vec{v}_1 + \vec{v}_2
\end{align*}
\]

Generic element of \( W \):

\[
a\vec{v}_1 + b\vec{v}_2 = \begin{pmatrix} a \\ b \\ a+b \end{pmatrix}, \quad a, b \in \mathbb{R}
\]

\( W \) = plane \( z = x+y \).

**Example**

**Def** If \( A \) is an \( m \times n \) matrix, then

\[
\{ \vec{x} \in \mathbb{R}^n \text{ such that } A\vec{x} = \vec{0} \}
\]

is called the **null space of \( A \)**.
null space of $A$

We'll see that null space of $A$ is always a subspace of $\mathbb{R}^n$.

Specific example

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$$

Solve for $A\vec{x} = \vec{0}$.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Row reducing...

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_1 = 5 + 2t$$
$$x_2 = -2s - 3t$$
$$x_3 = s$$
$$x_4 = t$$

Null space of $A$ = all vectors of form $$(5 + 2t, -2s - 3t, s, t)$$

Linear combination of

$$\begin{pmatrix} 1 \\ -2 \\ -3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

We've seen that subspaces can be expressed as all linear combinations of some fixed set of vectors.
4.4 - Span

**Def** If $V$ is a vector space and $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is a subset of $V$, then $\text{span}(S)$ is the collection of all linear combinations of the vectors in $S$.

**Theorem 4.4** Let $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be a subset of vector space $V$, then $\text{span}(S)$ is a subspace of $V$.

**Proof** Check closed under addition + scalar multi.

**add:** Let $\vec{u} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k$ be in $\text{span}(S)$, $\vec{v} = b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_k\vec{v}_k$ be in $\text{span}(S)$.

Then $\vec{u} + \vec{v} = (a_1 + b_1)\vec{v}_1 + (a_2 + b_2)\vec{v}_2 + \cdots + (a_k + b_k)\vec{v}_k$ is in $\text{span}(S)$.

**Scalar multi.** Let $\vec{u} = a_1\vec{v}_1 + \cdots + a_k\vec{v}_k$ be in $\text{span}(S)$ and $c \in \mathbb{R}$.

Then $c\vec{u} = ca_1\vec{v}_1 + ca_2\vec{v}_2 + \cdots + ca_k\vec{v}_k$ is in $\text{span}(S)$.

**Examples**

1. $\text{span}\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\} = \{(x, y) : x = x + y\}$ (the plane $z = x + y$)

2. Null space of $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$ is $\text{span}\{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}\}$.

3. In vector space $V = M_{22}$
3. In vector space \( V = M_{22} \)

\[ \text{span} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \]

What is this subspace? All \( 2 \times 2 \) matrices of the form

\[ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \]

So \( \text{span} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} = \) all \( 2 \times 2 \) diagonal matrices.

4. In vector space \( P_2 = \) polynomials of degree \( \leq 2 \),

\[ at^2 + bt + c \]

\[ \begin{align*}
\vec{v}_1 &= t^2 + 5t - 2 \\
\vec{v}_2 &= t^2 - 1
\end{align*} \]

\( W = \text{span} \left\{ t^2 + 5t - 2, t^2 - 1 \right\} \) is a subspace of \( P_2 \),

all polynomials of form

\[ a(t^2 + 5t - 2) + b(t^2 - 1) = (a+b)t^2 + 5at + (-2a-b) \]

Question: is \( W = P_2 \) or is \( W \) a proper subset of \( P_2 \).

Is \( t^2 + t + 1 \) in \( \text{span} \left\{ t^2 + 5t - 2, t^2 - 1 \right\} \)?

Can I find \( a, b \) with

\[ t^2 + t + 1 = (a+b)t^2 + 5at + (-2a-b) \]

\[ \begin{align*}
a+b &= 1 \\
5a &= 1 \\
-2a-b &= 1
\end{align*} \]

\[ \begin{pmatrix} a+b & 1 & 1 \\ 5a & 0 & 1 \\ -2a-b & -2 & 1 \end{pmatrix} \]
\[ a = \frac{1}{5} \text{ so } b = \frac{4}{5}, \text{ but then } -2\left(\frac{1}{5}\right) - \left(\frac{4}{5}\right) = -\frac{6}{5} \neq 1. \]

No solution, thus

\[ 1 + t + t^2 \text{ is not in span } \{t^2 + 5t - 2, t^2 - 1, 3\}. \]
Exam #1 (Chapter 1 - 3)

Draft (so far)

(55%) ≈ 10 Multiple choice problems (No partial credit)
(45%) ≈ 3 problems where show your work. (Partial credit)

Some MC problems are conceptual rather than computational.