Joint Parameter Estimation of the Ornstein-Uhlenbeck SDE driven by Fractional Brownian Motion

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Introduction

- Main Objective: To study the GMM (Generalized Method of Moments) joint estimator of the drift and memory parameters of the Ornstein-Uhlenbeck SDE driven by fBm.
- Joint work with Prof. Frederi Viens (Department of Statistics, Purdue University).

Outline

Outline

Fractional Brownian Motion

- The fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$ is the centered Gaussian process B_t^H, continuous a.s. with:

$$\mathrm{Cov}(B_t^H, B_s^H) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \qquad t, s \in \mathbb{R}.$$

- Some properties:
  - Self-Similarity: for each $a > 0$ $B_{at}^H \overset{d}{=} a^H B_t^H$.
  - It admits an integral representation with respect to the standard Brownian motion over a finite interval:

$$B_t^H = \int_0^t K_H(t,s)dW(s)$$

Fractional Gaussian noise (fGn)

- Definition:

$$N_t^H := B_t^H - B_{t-\alpha}^H$$

  where $\alpha > 0$.
- Gaussian and stationary process.
- Long-memory behavior of the increments when $H > \frac{1}{2}$ (in the sense that $\sum_n \rho(n) = \infty$).
- Ergodicity.

Fractional Gaussian noise (fGn)

- Autocovariance function:

$$\rho_\theta(t) = \frac{1}{2}\left[|t+\alpha|^{2H} + |t-\alpha|^{2H} - 2|t|^{2H}\right]$$

- Spectral density:

$$f_\theta(t) = 2c_H(1-\cos t)|t|^{-1-2H}$$

where $c_H = \frac{\sin(\pi H)\Gamma(2H+1)}{2\pi}$. (Beran, 1994).

Estimation of H (fBm and fGn)

Classical methods:

- R/S statistic (Hurst 1951), Variance plot (Heuristic method).
- Robinson (1992,1995): Semiparametric Gaussian estimation. (spectral information)

MLE Methods:

- Whittle's estimator.

Estimation of H

Methods using variations (filters):

- Coeurjolly (2001): consistent estimator of $H \in (0,1)$ based on the asymptotic behavior of discrete variations of the fBm. Asymptotic normality for $H < 3/4$.
- Tudor and Viens (2008): proved consistency and asymptotics of the second-order variational estimator (convergence to Rosenblatt random variable when $H > 3/4$) using Malliavin calculus.

Ornstein-Uhlenbeck SDE driven by fBm

- Cheridito (2003) Take $\lambda, \sigma > 0$ and ζ a.s bounded r.v. The Langevin equation:

$$X_t = \zeta - \lambda \int_0^t X_s ds + \sigma B_t^H, \qquad t \geq 0$$

has as an unique strong solution and it is called the fractional Ornstein-Uhlenbeck process:

$${}^{\zeta}X_t = e^{-\lambda t} \left(\zeta + \sigma \int_0^t e^{\lambda u} dB_u^H \right), \qquad t \leq T$$

and this integral exists in the Riemann-Stieltjes sense.

Properties

Cheridito (2003):

- Stationary solution (fOU process):

$$X_t = \sigma \int_{-\infty}^{t} e^{-\lambda(t-u)} dB_u^H, \qquad t > 0$$

- Autocovariance function (Pipiras and Taqqu, 2000):

$$\rho_\theta(t) = 2\sigma^2 c_H \int_0^\infty \cos(tx) \frac{x^{1-2H}}{\lambda^2 + x^2} dx$$

where $c_H = \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi}$.

Properties

Cheridito (2003):

- X_t has long memory when $H > \frac{1}{2}$, due to the following approximation when x is large:

$$\rho_\theta(x) = \frac{H(2H-1)}{\lambda} x^{2H-2} + O(x^{2H-4}).$$

- X_t is ergodic.
- X_t is not self-similar, but it exhibits asymptotic selfsimilarity (Bonami and Estrade, 2003):

$$f_\theta(x) = c_H |x|^{-1-2H} + O(|x|^{-3-2H}).$$

Estimation of λ given H (OU-fBm)

MLE estimators:

- Kleptsyna and Le Breton (2002):
  - MLE estimator based on Girsanov formula for fBm.
  - Strong consistency when $H > 1/2$.
- Tudor and Viens (2006):
  - Extended the K&L result to more general drift conditions.
  - Strong consistency of MLE estimator when $H < \frac{1}{2}$ using Malliavin calculus.

They work with the non-stationary case.

Estimation of λ given H (Least-Squares methods)

Hu and Nualart (2010)

- Estimate of λ which is strongly consistent for $H \geq \frac{1}{2}$.

$$\tilde{\lambda}_T = \left(\frac{1}{H\Gamma(2H)T} \int_0^T {}^0X_t^2 dt \right)^{-\frac{1}{2H}}$$

- $\tilde{\lambda}_T$ is asymptotically normal if $H \in (\frac{1}{2}, \frac{3}{4})$
- The proofs of these results rely mostly on Malliavin calculus techniques.

Estimation of H and λ

Methods based on variations:

- Biermé et al (2011): For fixed T, they use the results in Biermé and Richard (2006) to prove consistency and asymptotic normality of the joint estimator of (H, σ^2) for any **stationary** gaussian process with asymptotic self-similarity (Infill-asymptotics case).
- Brouste and Iacus (2012): For $T \to \infty$ and $\alpha \to 0$ they proved consistency and asymp. normality when $\frac{1}{2} < H < \frac{3}{4}$ for the pair (H, σ^2) (**non-stationary case**).

Outline

Preliminaries

- X_t: real-valued centered gaussian stationary process with spectral density $f_{\theta_0}(x)$.
- $f_\theta(x)$: continuous function with respect to x, continuously differentiable with respect to θ.
- θ belongs a compact set $\Theta \subset \mathbb{R}^p$.
- Bochner's theorem:

$$\rho_\theta(s) := \mathrm{Cov}(X_{t+s}, X_t) = \int_{\mathbb{R}} \cos(sx) f_\theta(x) dx$$

- If $\rho_\theta(s)$ is a continuous function of s, then the process X_t is ergodic.

Assumption 1

Take $\alpha > 0$ and L a positive integer. Then there exists $k \in \{0, 1, \ldots, L\}$ such that $\rho_\theta(\alpha k)$ is an injective function of θ.

Preliminaries

- Recall: $\mathbf{a}(l) := (a_0(l), \ldots, a_L(l))$ is a discrete filter of length $L+1$ and order l for $L \in \mathbb{Z}^+$ and $l \in \{0, \ldots, L\}$ if

$$\sum_{k=0}^{L} a_k(l) k^p = 0 \qquad \text{for } 0 \le p \le l-1$$

$$\sum_{k=0}^{L} a_k(l) k^p \neq 0 \qquad \text{if } p = l.$$

- Examples: finite-difference filters, Daubechies filters (wavelets).
- Assume that we can choose L filters with orders $l_i \in \{1, \ldots, L\}$ for $i = 1, \ldots, L$ and a extra filter with $l_0 = 0$.

Preliminaries

- Define the filtered process of order l_i and step size $\Delta_f > 0$ at $t \geq 0$ as:

$$\varphi_i(X_t) := \sum_{q=0}^{L} a_q(l_i) X_{t-\Delta_f q}$$

- Its expected value is: $V_i(\theta_0) := E[\varphi_i(X_t)^2] = \sum_{k=0}^{L} b_k(l_i) \rho_{\theta_0}(\Delta_f k)$.
- Define the set of moment equations by:

$$\mathbf{g}(X_t, \theta) := (g_0(X_t, \theta), \ldots, g_L(X_t, \theta))'$$

where

$$g_i(X_t, \theta) = \varphi_i(X_t)^2 - V_i(\theta), \qquad \text{for } 0 \leq i \leq L.$$

Preliminaries

- Assume that we have observed the stationary process X_t at times $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$ and $\alpha := t_i - t_{i-1} > 0$ (**fixed**).
- Assume there exists a sequence of symmetric positive-definite random matrices $\{\hat{A}_N\}$ such that $\hat{A}_N \xrightarrow{p} A$, and $A > 0$.
- Define:
  - $\hat{\mathbf{g}}_N(\theta) := \frac{1}{N-L+1} \sum_{i=L}^{N} \mathbf{g}(X_{t_i}, \theta)$. (sample moments)
  - $\hat{Q}_N(\theta) := \hat{\mathbf{g}}_N(\theta)' \hat{A}_N \hat{\mathbf{g}}_N(\theta)$.
  - $Q_0(\theta) := E[\mathbf{g}(X_t, \theta)]' A E[\mathbf{g}(X_t, \theta)]$.

GMM estimation

- Define the GMM estimator of θ_0:

$$\hat{\theta}_N := \text{argmin}_{\theta \in \Theta} \hat{Q}_N(\theta)$$
$$= \text{argmin}_{\theta \in \Theta} \left(\frac{1}{N} \sum_{i=1}^{N} \mathbf{g}(X_{t_i}, \theta) \right)^T \hat{A}_N \left(\frac{1}{N} \sum_{i=1}^{N} \mathbf{g}(X_{t_i}, \theta) \right).$$

Consistency

Lemma 1

Under the above assumptions:

(i)

$$\sup_{\theta \in \Theta} |\hat{Q}_N(\theta) - Q_0(\theta)| \overset{a.s}{\to} 0.$$

(ii)

$$Q_0(\theta) = 0.$$

if and only if $\theta = \theta_0$.

Consistency

Key aspects of proof:

- Ergodicity of X_t
- Continuity of $\rho_\theta(\cdot)$ over the compact set Θ.
- Injectivity of:

$$\boldsymbol{\rho}_\theta(\alpha) := \begin{bmatrix} \rho_\theta(\alpha \cdot 0) \\ \vdots \\ \rho_\theta(\alpha \cdot L) \end{bmatrix}$$

- Results of Newey and McFadden, 1994. (GMM)

We have all the conditions to apply:

Theorem 1 (Newey and McFadden, 1994)

Under the above assumptions, it holds that:

$$\hat{\theta}_N \xrightarrow{a.s.} \theta_0.$$

Asymptotic Normality of the sample moments

- Denote:

$$\hat{G}_N(\theta) := \nabla_\theta \hat{\mathbf{g}}_N(\theta) \qquad G(\theta) := E[\nabla_\theta \mathbf{g}(X_t, \theta)]$$

Assumption 2

For fixed $\alpha > 0$, assume that the $(L+1) \times p$ matrix $\nabla_\theta \boldsymbol{\rho}_\theta(\alpha)$ is a full-column rank matrix for any $\theta \in \Theta$.

- We can prove that: $\hat{G}_N(\theta) = G(\theta) = -\nabla_\theta V(\theta)$, where $V(\theta) = (V_0(\theta), \ldots, V_L(\theta))$.

Asymptotic Normality of the sample moments

And based on Biermé et al (2011) article, let us assume:

Assumption 3 (Biermé et al's condition)

Assume that for any $l, l' \in \{l_0, \ldots, l_L\}$ we have that:

$$R(u|l, l') := \min\left(1, |u|^{l+l'}\right) \sum_{p \in \mathbb{Z}} f_{\theta_0}\left(\frac{u + 2\pi p}{\alpha}\right) \in L^2((-\pi, \pi))$$

Lemma 2

Under Assumptions 1-3

$$\sqrt{N}\hat{\mathbf{g}}_N(\theta_0) \xrightarrow{d} N(0, \Omega)$$

where

$$\Omega_{ij} = 2\alpha^{-2} \int_{-\pi}^{\pi} |P_{a_{l_i}}[\cos u]|^2 \cdot |P_{a_{l_j}}[\cos u]|^2 \bar{f}_{\theta_0}(u/\alpha)^2 du$$

and $\bar{f}_{\theta_0}(u) := \sum_{p \in \mathbb{Z}} f_{\theta_0}(u + 2\pi p)$.

Note: $P_{a_l}(x) := \sum_{k=0}^{L} a_k(l) x^k$.

Sketch of proof

- Let $V_D(\theta_0) := \text{Diag}\left(1/V_j(\theta_0)\right)_{j\in\{0,\ldots,L\}}$. We scale the vector $\mathbf{g}(X_t, \theta_0)$ as follows:

$$\sqrt{N} V_D(\theta_0)\hat{\mathbf{g}}_N(\theta_0) = \frac{\sqrt{N}}{N-L+1}\sum_{i=L}^{N}\left(H_2(Z_{l_j,t_i})\right)_{j\in\{0,\ldots,L\}}$$

  where $Z_{l_j,t_i} := \frac{\varphi_j(X_{t_i})}{\sqrt{V_j(\theta_0)}}$ and $H_2(\cdot)$ is the 2nd-order Hermite process.
- Use the vector-valued version of the Breuer-Major theorem with spectral-information conditions (Biermé et al, 2011) to deduce the asymptotic behavior of the previous sum.

Asymptotic behavior of the error

- Let $\varepsilon_N := \hat{\theta}_N - \theta_0$. Using the mean value theorem:

$$\varepsilon_N := \hat{\theta}_N - \theta_0 = -\psi_N(\bar{\theta}_N, \hat{\theta}_N) \cdot \hat{\mathbf{g}}_N(\theta_0)$$

  where $\psi_N(\bar{\theta}_N, \hat{\theta}_N) := [G(\hat{\theta}_N)'\hat{A}_N G(\bar{\theta}_N)]^{-1} \cdot G(\hat{\theta}_N)'\hat{A}_N$.
- Also note that:

$$\psi_N(\bar{\theta}_N, \hat{\theta}_N) \xrightarrow{p} [G(\theta)'AG(\theta)]^{-1} G(\theta)'A$$

  then we can bound $E[\|\psi_N(\bar{\theta}_N, \hat{\theta}_N)\|^{4p}]$ for any $p > 0$

Asymptotic behavior of the error

- X_t can be represented as a Wiener-Ito integral with respect to the standard Brownian motion:

$$X_{t_k} = I_1(A_k(\cdot|\theta_0))$$

where $A_k(x|\theta_0) := \cos(\alpha kx)\bar{f}_{\theta_0}(x)$.

- Using the multiplication rule of Wiener integrals:

$$(\hat{\mathbf{g}}_N(\theta))_i = I_2[B_{i,j}(\cdot|\theta)]$$

where $B_{i,j}(\cdot|\theta)$ is a kernel depending on the filter **a**.

Asymptotic behavior of the error

- We already proved a CLT for $\hat{\mathbf{g}}_N(\theta_0)$, hence, for all i:

$$E[|(\hat{g}_N(\theta_0))_i|^2] = O(N^{-1})$$

- Let $p > 0$, we can use the equivalence of L^p-norms of a fixed Weiner chaos to get:

$$E[\|\hat{\mathbf{g}}_N(\theta_0)\|^{4p}] < \frac{\mathcal{T}_{p,L}}{N^{2p}}$$

- By Borel-Cantelli, we conclude:

Asymptotic behavior of the error

Corollary 1

Under Assumptions 1-3 we have:

(i) $E[\|\hat{\theta}_N - \theta_0\|^2] = O(N^{-1})$

(ii) $N^\gamma \|\hat{\theta}_N - \theta_0\| \xrightarrow{a.s.} 0$ *for any* $\gamma < \frac{1}{2}$.

Asymptotic behavior of the error

And we can generalize the previous result as follows:

Theorem 2

Let $\hat{\theta}_N$ the GMM estimator of θ_0. Assume there exists a diagonal $(L+1) \times (L+1)$ matrix $D_N(\theta_0)$ such that:

$$E[D_N^{(i,i)}(\theta_0)|(\hat{\mathbf{g}}_N(\theta_0))_i|^2] = O(1)$$

for all $i \in \{0, \dots, L\}$. Then:

(i) $E[\|\hat{\theta}_N - \theta_0\|^2] = O\left(\max_{0 \le i \le L}\{D_N^{(i,i)}(\theta_0)\}\right)$

(ii) *If* $\max_{0 \le i \le L}\{D_N^{(i,i)}(\theta_0)\} = \frac{f(N)}{N^\nu}$, *for $f(N) = o(N)$ and $\nu > 0$ then for any $\gamma < \frac{\nu}{2}$:*

$$N^\gamma \|\varepsilon_N\| \xrightarrow{a.s.} 0.$$

Asymptotic Normality

Theorem 3

Let X_t a Gaussian stationary process with parameter θ_0. If $\hat{\theta}_N$ is the GMM estimator of θ_0, then under Assumptions 1-3, it holds that:

$$\sqrt{N}(\hat{\theta}_N - \theta_0) \xrightarrow{d} N(0, \mathcal{C}(\theta_0)\Omega\mathcal{C}(\theta_0)')$$

where $\mathcal{C}(\theta_0) = \left[G(\theta_0)' A G(\theta_0)\right]^{-1} G(\theta_0)' A$.

Key ideas of the proof (Newey and McFadden, 1994; Hansen, 1982):

- Linear behavior of $\varepsilon_N = \hat{\theta}_N - \theta_0$.
- Slutsky's theorem.

Efficiency

- The GMM asymptotic variance is minimized by taking $A = \Omega^{-1}$.
- There are numerical techniques to compute $\hat{A}_N$ such that $\hat{A}_N \xrightarrow{p} \Omega^{-1}$, for example (Hansen et al., 1996):
  - Two-step estimators.
  - Iterative estimator.
  - Continuous-updating estimator.

Outline

Preliminaries

- Assume that there exist a closed rectangle $\Theta \subset \mathbb{R}^2$ such that $\theta = (H, \lambda) \in \Theta$, and assume that $\sigma = 1$.
- $\rho_\theta(t)$ and its partial derivatives are continuous functions of θ.
- The Assumptions 1 and 2 are difficult to confirm due to the analytical complexity of the covariance function $\rho_\theta(t)$. we decided to check these two assumptions at least locally, by checking numerically that:

$$\det \begin{bmatrix} \frac{\partial \rho_\theta(0)}{\partial H} & \frac{\partial \rho_\theta(0)}{\partial \lambda} \\ \frac{\partial \rho_\theta(\alpha)}{\partial H} & \frac{\partial \rho_\theta(\alpha)}{\partial \lambda} \end{bmatrix} \neq 0$$

  for different values of θ and α.
- The injectivity approximately holds for $0 < \lambda < 5$ and $H > 0.3$.

Lemma 3

Let X_t be the stationary fOU process with parameters $\theta = (H, \lambda)$. The Assumption 3 (Biermé et al's condition) holds under the following two cases:

Case 1 *If $l + l' > 1$ then it holds for all $H \in (0, 1)$.*

Case 2 *If $l + l' \leq 1$ then it holds if $H \in (0, \frac{3}{4})$.*

Conclusion: Assumption 3 is not valid for the first component of $\hat{\mathbf{g}}_N$ if $H \geq \frac{3}{4}$.

Asymptotic Normality of the sample errors

Using the previous lemma, for $H < \frac{3}{4}$:

$$\sqrt{N}\hat{\mathbf{g}}_N(\theta) \xrightarrow{d} N(0, \Lambda(\theta))$$

where the covariance matrix $\Lambda(\theta)$ has entries:

$$\Lambda_{ij}(\theta) = 2c_H^2 \alpha^{4H} \underbrace{\int_{-\pi}^{\pi} \left| P_{a_{l_i}}[\cos u] \right|^2 \left| P_{a_{l_j}}[\cos u] \right|^2 \left[\sum_{p \in \mathbb{Z}} \frac{|u + 2\pi p|^{1-2H}}{(u + 2\pi p)^2 + (\lambda\alpha)^2} \right]^2 du}_{\mathcal{K}_{i,j}(\alpha)}$$

Asymptotic Normality of the sample errors

Lemma 4

Assume that X_t is a stationary fOU process with parameters $\theta = (H, \lambda)$ where $H \geq \frac{3}{4}$. Then:

(i) *If $H = \frac{3}{4}$:*

$$\sqrt{\frac{N}{\log N}} (\hat{\mathbf{g}}_N(\theta))_0 \xrightarrow{d} N(0, 2\alpha^{-1} c_\theta^2)$$

where $c_\theta = \frac{H(2H-1)}{\lambda} = \frac{3}{8\lambda}$.

(ii) *If $H > \frac{3}{4}$, $(\hat{\mathbf{g}}_N(\theta))_0$ does not converge to a normal law, or even a second-chaos law. However,*

$$E\left[\left| N^{2-2H} (\hat{\mathbf{g}}_N(\theta))_0 \right|^2 \right] = O(1).$$

Sketch of proof

- Denote: $\hat{g}_{N,0}(\theta) = (\hat{\mathbf{g}}_N(\theta))_0$.
- For $H = \frac{3}{4}$, as $N \to \infty$:

$$E\left[\frac{N}{\log N}|\hat{g}_{N,0}(\theta)|^2\right] \longrightarrow 2\alpha^{-1}c_\theta^2 := \tilde{c}_1$$

  where $c_\theta := \frac{H(2H-1)}{\lambda}$.
- For $H > \frac{3}{4}$:

$$E[|N^{2-2H}\hat{g}_{N,0}(\theta)|^2] \longrightarrow \frac{2\alpha^{4H-4}c_\theta^2}{(2H-1)(4H-3)} := \tilde{c}_2$$

Sketch of proof

- If $F_N = \begin{cases} \sqrt{\frac{N}{\tilde{c}_1 \log N}} \hat{g}_{N,0}(\theta) & \text{if } H = \frac{3}{4} \\ \sqrt{\frac{N^{4-4H}}{\tilde{c}_2}} \hat{g}_{N,0}(\theta) & \text{if } H > \frac{3}{4} \end{cases}$ then we need to prove

$$\|DF_N\|_{\mathcal{H}}^2 \xrightarrow{L^2(\Omega)} 2$$

  where $\mathcal{H} = L^2((-\pi, \pi))$.
- This result is valid only if $H = \frac{3}{4}$.
- Hence we can use Nualart and Ortiz-Latorre (2008) to conclude asymptotic normality of F_N.

Sketch of proof

- In the case $H > \frac{3}{4}$ we can use the same criteria to conclude that F_N does not have a normal limit in law.
- Furthermore, F_N has this kernel:

$$\mathcal{I}_N(r,s) := \frac{N^{1-2H}}{\sqrt{\tilde{c}_2}} \sum_{j=1}^{N} (A_j \otimes A_j)(r,s)$$

- It can be proved that $\mathcal{I}_N(r,s)$ is not a Cauchy sequence in $\mathcal{H}^2$. Then F_N does not have a 2nd-chaos limit.

Asymptotic behavior of the error

Proposition 1

Let X_t be a fOU process with parameters $\theta = (H, \lambda)$. Then the GMM estimate $\hat{\theta}_N$ satisfies:

(i)

$$E[\|\hat{\theta}_N - \theta\|^2] = \begin{cases} O(N^{-1}) & \text{if } H \in (0, \frac{3}{4}) \\ O\left(\frac{\log N}{N}\right) & \text{if } H = \frac{3}{4} \\ O(N^{4H-4}) & \text{if } H \in (\frac{3}{4}, 1) \end{cases}$$

(ii) *As N goes to infinity:*

$$N^{\gamma} \|\hat{\theta}_N - \theta\| \xrightarrow{a.s.} 0$$

for all $\gamma < \frac{1}{2}$ (if $H \leq \frac{3}{4}$). Otherwise for all $\gamma < 2 - 2H$, if $H > \frac{3}{4}$.

Asymptotics of $\hat{\theta}_N$:

Proposition 2

Let X_t be the stationary fOU process with parameters $\theta = (H, \lambda)$. Then, for any positive-definite matrix A, the GMM estimator of θ is consistent for any $H \in (0,1)$ and:

- *If $H \in (0, \frac{3}{4})$:*

$$\sqrt{N}(\hat{\theta}_N - \theta) \xrightarrow{d} N(0, \mathcal{C}(\theta)\Lambda\mathcal{C}(\theta)')$$

  where $\mathcal{C}(\theta) = [G(\theta)'AG(\theta)]^{-1}G(\theta)'A$ and $\Lambda = 2c_H^2\alpha^{4H}\mathcal{K}(\alpha)$.

- *If $H = \frac{3}{4}$, then*

$$\sqrt{\frac{N}{\log N}}(\hat{\theta}_N - \theta) \xrightarrow{d} N\left(0, \frac{2c_\theta^2}{\alpha}(\mathcal{C}(\theta))_1'(\mathcal{C}(\theta))_1\right)$$

  where $(\mathcal{C}(\theta))_1$ is the first column of $\mathcal{C}(\theta)$.

Asymptotics of $\hat{\theta}_N$:

- If $H > \frac{3}{4}$, the GMM estimator does not converge to a multivariate normal law or even a second-chaos law.

Outline

Numerical Details

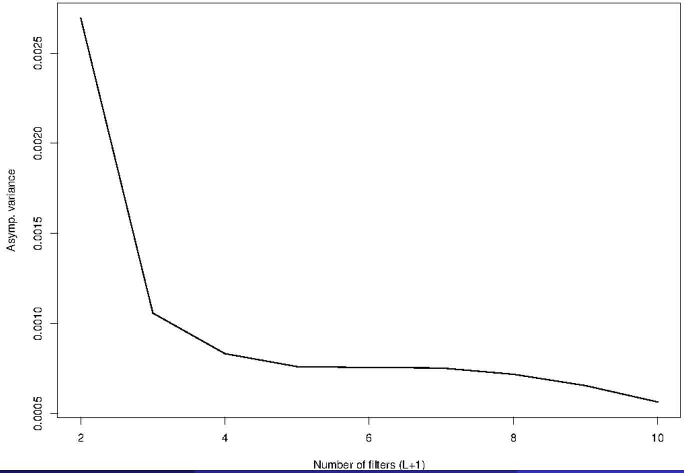
- Approximation of $\rho_\theta(t) = 2\sigma^2 c_H \int_0^\infty \cos(tx) \frac{x^{1-2H}}{\lambda^2 + x^2} dx$:
  - Filon's Method (integration of oscillatory integrands)
  - 4-5 precision digits.
- Simulation of fBm: Davies and Harte (1987).
- Approximation of fBm-OU process: the stationary fBm-OU satisfies:

$$X_{t_{i+1}} = e^{-\lambda\Delta} X_{t_i} + \sigma \int_{t_i}^{t_{i+1}} e^{-\lambda(t_i - u)} dB_u^H$$

  where $X_0 \sim N(0, \rho_\theta(0))$.
- For comparison purposes we use the `yuima` R package of Brouste and Iacus (2012).
- Finite-difference filters.

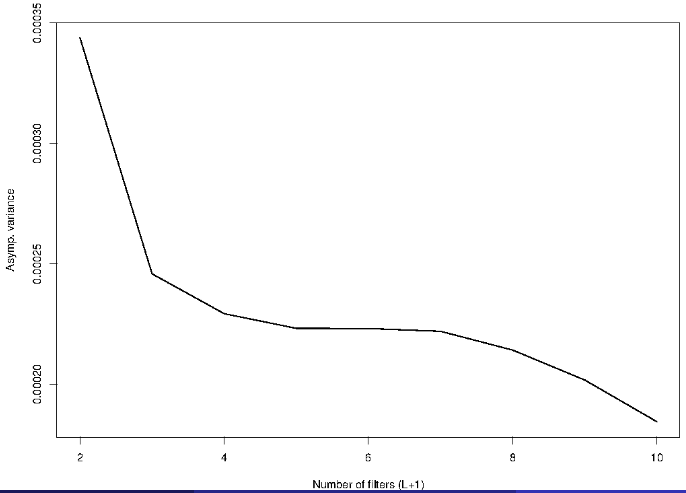
Asymptotic behavior of $\sqrt{N}(\hat{\theta}_N - \theta)$

Figure: Asymptotic variance of $\sqrt{N}(\hat{H}_N - 0.62)$



Asymptotic behavior of $\sqrt{N}(\hat{\theta}_N - \theta)$

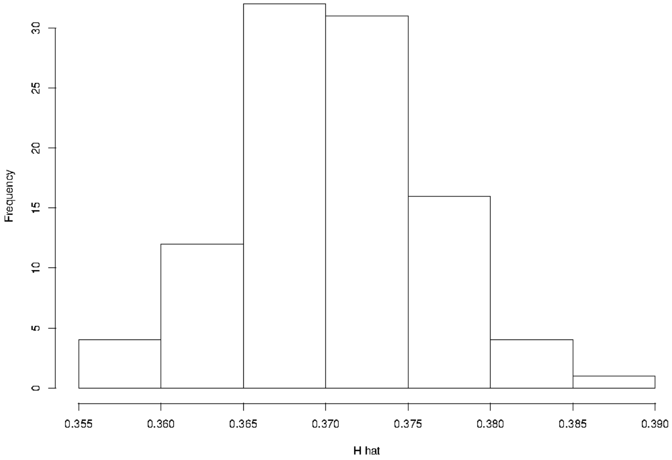
Figure: Asymptotic variance of $\sqrt{N}(\hat{\lambda}_N - 0.8)$



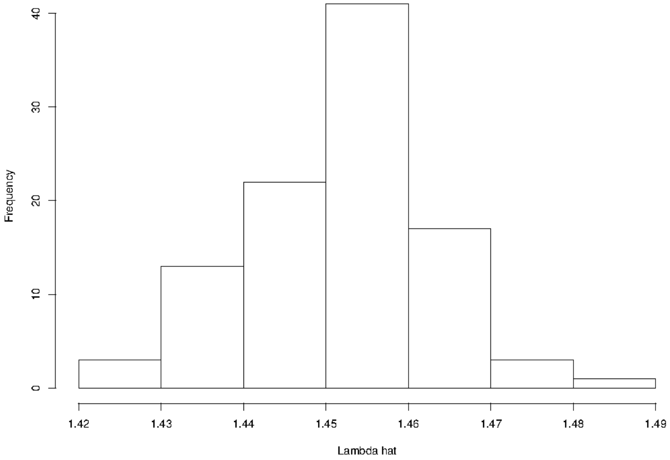
Numerical Details

- $(H, \lambda) = (0.37, 1.45)$
- $N = 1000$, $\alpha = 1$.
- $L = 3$ (filters' maximum order)
- Number of repetitions in the sampling dist.: 100.
- Efficient matrix estimation: Continuous-updating estimator.

Sampling distribution of $\hat{H}_N$



Sampling distribution of $\hat{\lambda}_N$

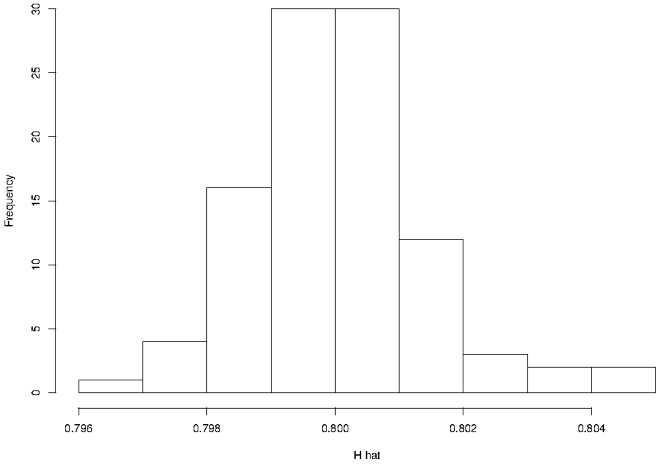


Some statistics...

Empirical MSE of $\hat{\theta}_N$	0.000155
Asymp. variance of $\hat{H}_N$ (empirical)	3.11×10^{-5}
Asymp. variance of $\hat{H}_N$ (theoretical)	4.96×10^{-5}
Asymp. variance of $\hat{\lambda}_N$ (empirical)	0.000122
Asymp. variance of $\hat{\lambda}_N$ (theoretical)	0.000193

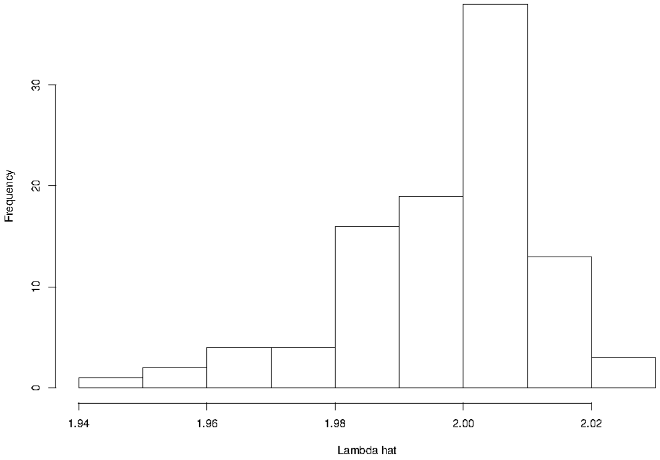
Sampling distribution of $\hat{H}_N$

$(H, \lambda) = (0.8, 2)$ with 4 filters.



Sampling distribution of $\hat{\lambda}_N$

$(H, \lambda) = (0.8, 2)$ with 4 filters.



Future Work

- More accurate simulation of the fOU process?
- Asymptotic law of $\hat{\theta}_N$ when $H > \frac{3}{4}$? (fOU case)

Thanks!