On the chaotic character of some parabolic SPDEs

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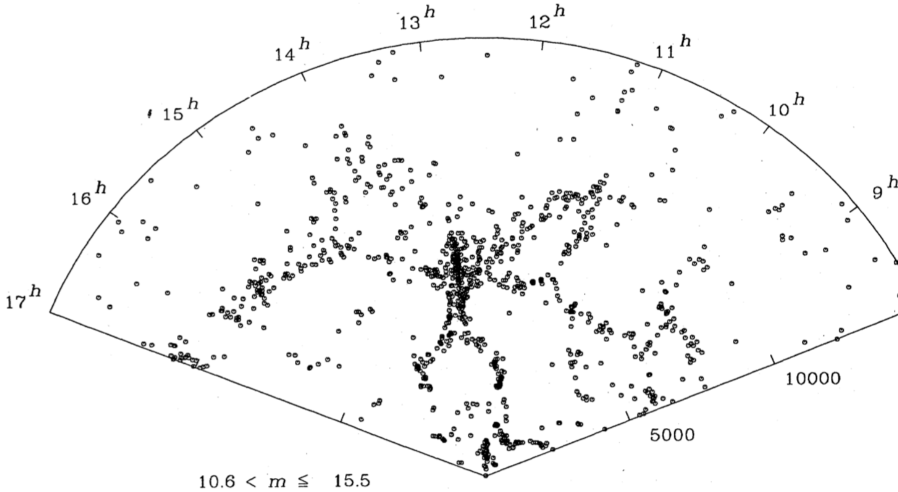
Large-scale distribution of galaxies

S. F. Shandarin and Ya. B. Zeldovitch, *Rev. Modern Phys.* **61**(2) (1989) 185–220



A simple model for intermittency

(Zeldovich–Ruzmaikin–Sokoloff, 1990)

- Intermittency occurs when we multiply many roughly-independent r.v.'s ; e.g., $\xi_1, \xi_2, \ldots$ i.i.d. with $P\{\xi_1 = 2\} = P\{\xi_1 = 0\} = 1/2$
- Then

$$u_n := \prod_{j=1}^{n} \xi_j = \begin{cases} 2^n & \text{with probab. } 2^{-n}, \\ 0 & \text{with probab. } 1 - 2^{-n}. \end{cases}$$

- Conclusions:
  - $u_n = 0$ for all n large [a.s.]; in particular, $u_n \to 0$ a.s.
  - $n^{-1} \log E(u_n^k) \to \gamma_k := (k-1)\log 2$ for all $k > 1$
- Now replicate this experiment
- Is this degeneracy because of the many zeros? No

A second simple model for intermittency

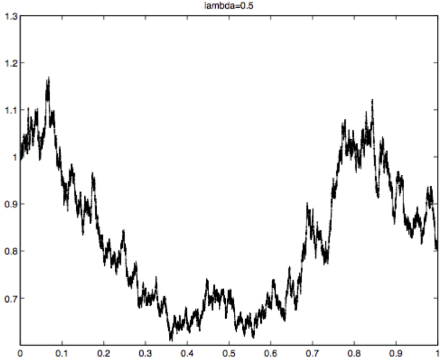
(Zeldovich–Ruzmaikin–Sokoloff, 1990)

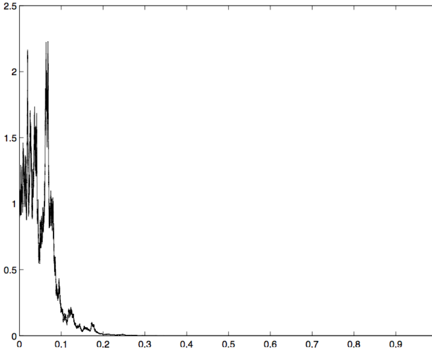
- Let b denote 1-D Brownian motion and consider the exponential martingale $u_t := e^{\lambda b_t - (\lambda^2 t/2)}$
- $u_t \to 0$ as $t \to \infty$ [strong law]
- $t^{-1} \log E(u_t^k) = \frac{1}{2}\lambda^2 k(k-1) \to \gamma_k := \frac{1}{2}\lambda^2 k(k-1)$ for $k > 1$
- In the first example, $\gamma_k \approx k \log 2$; in the second, $\gamma_k \approx \frac{1}{2}\lambda^2 k^2$
- The examples are "similar,"

$$e^{b_t - (t/2)} \approx \prod_j \left(1 - (\Delta b)_j - \frac{1}{2}(\Delta t)_j\right)$$

A simulation $[\dot{u}_t(x) = (1/2)u_t''(x) + \lambda u_t(x)\eta_t,\ u_0 \equiv 1]$

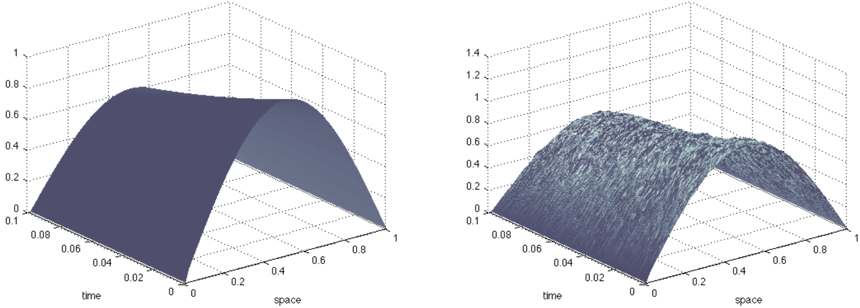
$u_t = \exp\{\lambda b_t - (\lambda^2 t/2)\}$ with $\lambda = 0.5$ (left) and $\lambda = 5$ (right)





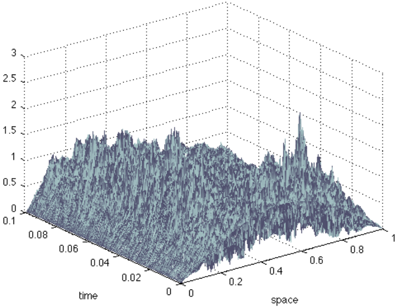
A simulation $[\dot{u}_t(x) = (1/2)u_t''(x) + \lambda u_t(x)\eta_t(x),$ $u_0(x) = \sin(\pi x),\ 0 \le x \le 1;\ u_t(0) = u_t(1) = 0.]$

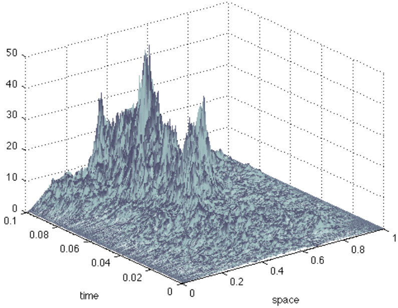
$\lambda = 0$ (left; $u_t(x) = \sin(\pi x)\exp(-\pi^2 t/2)$) and $\lambda = 0.1$ (right)



A simulation $[\dot{u}_t(x) = (1/2)u_t''(x) + \lambda u_t(x)\eta_t(x),$ $u_0(x) = \sin(\pi x),\ 0 \leq x \leq 1;\ u_t(0) = u_t(1) = 0.]$

$\lambda = 1.3$ (left) and $\lambda = 2$ (right)





The model

$$\frac{\partial}{\partial t} u_t(x) = \frac{\kappa}{2} \frac{\partial^2}{\partial x^2} u_t(x) + \sigma(u_t(x))\eta_t(x),$$

where:

1. $\kappa > 0$;
2. $\sigma : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous;
3. η is space-time white noise; i.e., a centered GGRF with

$$\mathrm{Cov}\left(\eta_t(x)\,, \eta_s(y)\right) = \delta_0(t-s)\delta_0(x-y)$$

4. $u_0 : \mathbf{R} \to \mathbf{R}_+$ nonrandom, bounded, and measurable;
5. u exists, is unique and continuous (Walsh, 1986)

The model

$\partial_t u = (\kappa/2)\partial_{xx} u + \sigma(u)\eta$

- Many physically-interesting choices of $\sigma \not\equiv 0$:
  - σ periodic/quasi-periodic/stationary process [random media];
  - $\sigma(u) \propto u$ [the parabolic Anderson model/KPZ/Br. Br. motion in random environment];
  - $\sigma(u) \propto \sqrt{u}$ [super processes];
  - $\sigma(u) \propto \sqrt{u(1-u)}$ [stoch. KPP]; ....
- Today, we will say a few things about the first two examples [where σ is Lipschitz]

Weak intermittency

$\partial_t u = (\kappa/2)\partial_{xx} u + \sigma(u)\eta$

- (weak) intermittency [Bertini–Cancrini, 1994; Carmona–Molchanov, 1994; Molchanov, 1991; Foondun–K., 2010; Zel'dovitch et al, 1985, 1988, 1990; ...]:

$$0 < \limsup_{t\to\infty} \frac{1}{t} \log E\left(|u_t(x)|^k\right) < \infty \quad (k \geq 2, x \in \mathbf{R})$$

- Weak intermittency implies "localization" on large time scales.
- Physical intermittency is expected to hold because the SPDE is typically "chaotic," and for many choices of σ:
  - For all $t > 0$; and
  - both in time, *and* space
- Today: What happens before the onset of localization?

Optimal regularity

- Can frequently understand parabolic equations via optimal regularity [Lunardi, 1995, and older works by Pazy, Kato, ...]
- If $\sigma(0) = 0$, then the fact that $u_0(x) \geq 0$ implies that $u_t(x) \geq 0$ [Mueller's comparison principle]
- If $\sigma(0) = 0$ and $u_0 \in L^2(\mathbf{R})$ then $u_t \in L^2(\mathbf{R})$ a.s. (Dalang–Mueller, 2003, but likely known earlier also)
- If $u_0 \in C^\alpha(\mathbf{R})$ for some $\alpha > \frac{1}{2}$ and has compact support, and if $\sigma(0) = 0$, then $\sup_{x \in \mathbf{R}} u_t(x) < \infty$ a.s. for all $t > 0$ (Foondun–Kh, 2010+)
- Today's goal: The solution can be sensitive to the choice of u_0 (we study cases where u_t is unbounded for all $t > 0$)

A reduction

- $\dot{u} = (\kappa/2)u'' + \sigma(u)\eta$
- Suppose $\sigma(x_0) = 0$ for some $x_0 > 0$
- If $u_0(x) \leq x_0$ then $u_t(x) \leq x_0$ [Mueller's comparison theorem]
- Therefore, today we are interested only in the case that $\sigma(x) > 0$ for all $x > 0$
- We consider the case that $\inf_{x \in \mathbf{R}} u_0(x) > 0$ only

Theorem (Conus–Joseph–K)

A case of minimum noise

- $\dot{u} = (\kappa/2)u'' + \sigma(u)\eta$
- If $\inf_{x\in\mathbf{R}} \sigma(x) > 0$, then

$$\limsup_{|x|\to\infty} \frac{u_t(x)}{(\log|x|)^{1/6}} \geq \text{const}\cdot\kappa^{-1/12} \qquad \text{a.s. for all } t > 0$$

- $\exists$ weaker versions that allow mild decay for σ; e.g., suppose $\sigma(x) > 0$ for all $x \geq 0$ and $\exists\gamma \in (0\,,1/6)$ such that $\sigma(x) \gg (\log|x|)^{-(1/6)+\gamma}$. Then a.s. for all $t > 0$,

$$\limsup_{|x|\to\infty} \frac{u_t(x)}{(\log|x|)^{\gamma}} \geq \text{const}\cdot\kappa^{-1/12}.$$

Theorem (Conus–Joseph–K)

The moderately noisy case

- $\dot{u} = (\kappa/2)u'' + \sigma(u)\eta$
- If $0 < \inf_{x\geq 0} \sigma(x) \leq \sup_{x\geq 0} \sigma(x) < \infty$, then

$$\limsup_{|x|\to\infty} \frac{u_t(x)}{(\log|x|)^{1/2}} \asymp \kappa^{-1/4} \qquad \text{a.s. for all } t > 0$$

- Power of κ suggests the universality class of random walks in weak interactions with their random environment

Theorem (Conus–Joseph–K)

The parabolic Anderson case

- $\dot{u} = (\kappa/2)u'' + cu\eta \qquad [\sigma(x) = cx]$
- If $c > 0$, then

$$\limsup_{|x|\to\infty} \frac{\log u_t(x)}{(\log|x|)^{2/3}} \asymp \frac{1}{\kappa^{1/3}} \qquad \text{a.s. for all } t > 0$$

- $u_t(x) \approx \exp\{\text{const} \cdot (\log|x|/\sqrt{\kappa})^{2/3}\}$
- Power of κ suggests the universality class of random matrix models
- "scaling exponents" $(1/3\,, 2/3)$

A connection to KPZ

- The KPZ equation (1986): If $\lambda \in \mathbf{R}$ is fixed then

$$\dot{h} = \frac{\kappa}{2} h'' + \frac{\kappa\lambda}{2}(h')^2 + \eta$$

- Rigorous meaning (?): A formal Hopf–Cole transformation $[u_t(x) = \exp\{h_t(x)\}]$ yields

$$\dot{u} = \frac{\kappa}{2} u'' + u\eta$$

- "Therefore," if $h_t^*(x) := \sup_{|z| \le x} h_t(z)$, then

$$\frac{\text{const}}{\kappa^{1/3}} \le \liminf_{R \to \infty} \frac{h_t^*(e^R)}{R^{2/3}} \le \limsup_{|x| \to \infty} \frac{h_t^*(e^R)}{R^{2/3}} \le \frac{\text{const}}{\kappa^{1/3}} \qquad \text{a.s. for all } t >$$

- Related to recent results by Balázs–Quastel–Seppäläinen (2011) & Amir–Corwin–Quastel (2011)

Colored noise

$\dot{u}_t(x) = (\kappa/2)(\Delta u_t)(x) + \sigma(u_t(x))\eta_t(x)$, $t > 0$, $x \in \mathbf{R}^d$

- Now
$$\mathrm{Cov}\left(\eta_t(x)\,,\eta_s(y)\right) = \delta_0(s-t)f(x-y)$$
(Dalang, 1999; Hu–Nualart, 2009, …)
- Suppose $f = h * \tilde{h}$ for some $h \in L^2(\mathbf{R}^d)$, so $\exists$ a unique solution for all $d \geq 1$
- $\exists$ KPZ version also (Medina–Hwa–Kardar–Zhang, 1989)

Theorem (Conus–Joseph–K–Shiu)

The parabolic Anderson case

- $\dot{u} = (\kappa/2)\Delta u + cu\eta \qquad [\sigma(x) = cx]$
- If $c > 0$ and $\int_{\|z\|>N} |h(z)|^2\,dz = O(N^{-a})$ for some $a > 0$, then

$$\limsup_{|x|\to\infty} \frac{\log u_t(x)}{(\log|x|)^{1/2}} \asymp 1 \qquad \text{a.s. for all } t > 0 \text{ and } \kappa \text{ small}$$

- There are other variations as well
- "scaling exponents" $(0\,,1/2)$
- Are there in-between models? Yes.

Theorem (Conus–Joseph–K–Shiu)

The parabolic Anderson case

- $\dot{u} = (\kappa/2)\Delta u + cu\eta \qquad [\sigma(x) = cx]$
- $\mathrm{Cov}(\eta_t(x)\,, \eta_s(y)) = \delta_0(t-s)\cdot\|x-y\|^{-\alpha}$
- The solution ∃! when $\alpha < \min(d\,, 2)$ [Dalang, 1999]
- If $c > 0$, then

$$\limsup_{|x|\to\infty} \frac{\log u_t(x)}{(\log\|x\|)^{2/(4-\alpha)}} \asymp \kappa^{-\alpha/(4-\alpha)} \qquad \text{a.s. for all } t > 0$$

- "scaling exponents" $(2\psi - 1\,, \psi) = \left({}^{\alpha}/_{(4-\alpha)}\,, {}^{2}/_{(4-\alpha)}\right)$

  $\underbrace{[T + 1 = 2X]}$ KPZ relation
- $f = h * \tilde{h} \;\Leftrightarrow\; \alpha = 0$, and $f = \delta_0 \;\Leftrightarrow\; \alpha = 1 = \min(d\,, 2)$

  [spectral analogies]