Martingale transforms and their projection operators on manifolds

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Probability seminar

Based on a joint work with Rodrigo Bañuelos
To appear in Potential Analysis

Motivation

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where X_i^* denotes the formal adjoint of X_i with respect to μ and where $V : \mathbb{M} \to \mathbb{R}$ is a non-positive smooth function.

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We study the boundedness in L^p, $1 < p < \infty$ of the operator

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where $P_t = e^{tL}$ and $A(t, x)$ is a matrix of smooth bounded functions.

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where $\Psi_a(\lambda) = -2\lambda \int_0^\infty a(t) e^{-2\lambda t} dt$. This is a so-called multiplier of Laplace transform type.

Motivation

- If L is the Laplace-Beltrami operator on a Lie group G of compact type and if A is constant, then

$$\mathcal{S}_A f = \sum_{i,j} A_{ij} \left(\sum_{i=1}^{d} X_i^2 \right)^{-1} X_i X_j f \, .$$

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Defining then the Riesz transforms on G by

$$R_j f = \left(- \sum_{i=1}^{d} X_i^2 \right)^{-1/2} X_j f$$

we see that

$$\mathcal{S}_A f = \sum_{i,j=1}^{d} A_{ij} R_i R_j f .$$

Probabilistic representation of $\mathcal{S}_A$

The diffusion $(Y_t)_{t\geq 0}$ with generator $-\frac{1}{2}\sum_{i=1}^{d} X_i^* X_i$ can be constructed via the Stratonovitch stochastic differential equation

$$dY_t = X_0(Y_t)dt + \sum_{i=1}^{d} X_i(Y_t) \circ dB_t^i,$$

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The celebrated Feynman-Kac formula reads

$$P_t f(x) = \mathbb{E}_x \left(e^{\int_0^t V(Y_s)ds} f(Y_t) \right).$$

Probabilistic representation of $\mathcal{S}_A$

Theorem

In L^p, $1 < p < \infty$, *we have* $\lim_{T\to\infty} \mathcal{S}_A^T = \mathcal{S}_A$, *where*

$$\mathcal{S}_A^T f(x) = \mathbb{E}\left(e^{\int_0^T V(Y_s)ds} \int_0^T e^{-\int_0^t V(Y_s)ds} dM_t \mid Y_T = x \right).$$

and

$$dM_t = \sum_{i,j=1}^{d} A_{ij}(T-t, Y_t)(X_j P_{T-t} f)(Y_t) dB_t^i$$

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$$dM_t = \sum_{i,j=1}^{d} A_{ij}(T-t, Y_t)(X_j P_{T-t} f)(Y_t) dB_t^i$$

Since the conditional expectation is a contraction in L^p, we now essentially need to control the L^p norm of the stochastic integral

$$e^{\int_0^T V(Y_s)ds} \int_0^T e^{-\int_0^t V(Y_s)ds} dM_t$$

A variation of the BDG inequality

Theorem

Let $T > 0$ and $(M_t)_{0 \leq t \leq T}$ be a continuous local martingale. Consider the process

$$Z_t = e^{\int_0^t V_s ds} \int_0^t e^{-\int_0^s V_u du} dM_s,$$

where $(V_t)_{0 \leq t \leq T}$ is a non positive adapted and continuous process. For every $0 < p < \infty$, there is a universal constant C_p, independent of T, $(M_t)_{0 \leq t \leq T}$ and $(V_t)_{0 \leq t \leq T}$ such that

$$\mathbb{E}\left(\left(\sup_{0 \leq t \leq T} |Z_t|\right)^p\right) \leq C_p \mathbb{E}\left([M, M]_T^{\frac{p}{2}}\right).$$

Proof of the variation of the BDG inequality

By stopping it is enough to prove the result for bounded M. Let $q \geq 2$. We have

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and from Itô's formula we have

$$\begin{aligned} d|Z_t|^q &= q|Z_t|^{q-1}\mathbf{sgn}(Z_t)dZ_t + \frac{1}{2}q(q-1)|Z_t|^{q-2}d[M]_t \\ &= q|Z_t|^q V_t dt + q\mathbf{sgn}(Z_t)|Z_t|^{q-1}dM_t + \frac{1}{2}q(q-1)|Z_t|^{q-2}d[M]_t. \end{aligned}$$

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Since $V_t \leq 0$, as a consequence of the Doob's optional sampling theorem, we get that for every bounded stopping time τ,

$$\mathbb{E}\left(|Z_\tau|^q\right) \leq \frac{1}{2}q(q-1)\mathbb{E}\left(\int_0^\tau |Z_t|^{q-2}d[M]_t\right).$$

Lenglart's domination inequality

Theorem (Lenglart)

Let $(N_t)_{t\geq 0}$ *be a positive adapted right-continuous process and* $(A_t)_{t\geq 0}$ *be an increasing process. Assume that for every bounded stopping time* τ*,*

$$\mathbb{E}(N_\tau) \leq \mathbb{E}(A_\tau).$$

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Then, for every $k \in (0,1)$,

$$\mathbb{E}\left(\left(\sup_{0\leq t\leq T} N_t\right)^k\right) \leq \frac{2-k}{1-k}\mathbb{E}\left(A_T^k\right).$$

Proof of the variation of the BDG inequality

From the Lenglart's domination inequality, we deduce then that for every $k \in (0,1)$,

$$\mathbb{E}\left(\left(\sup_{0\leq t\leq T}|Z_t|^q\right)^k\right) \leq C_{k,q}\mathbb{E}\left(\left(\int_0^T |Z_t|^{q-2}d[M]_t\right)^k\right).$$

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From the Lenglart's domination inequality, we deduce then that for every $k \in (0,1)$,

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We finally compute

$$\begin{aligned}
&\mathbb{E}\left(\left(\int_0^T |Z_t|^{q-2}d[M]_t\right)^k\right)\\
\le&\mathbb{E}\left(\left(\sup_{0\le t\le T}|Z_t|\right)^{k(q-2)}\left(\int_0^T d[M]_t\right)^k\right)\\
\le&\mathbb{E}\left(\left(\sup_{0\le t\le T}|Z_t|\right)^{kq}\right)^{1-\frac{2}{q}}\mathbb{E}\left([M]_T^{\frac{kq}{2}}\right)^{\frac{2}{q}}.
\end{aligned}$$

Control of $\mathcal{S}_A^T$

Thanks to the previous result, we are let with the problem of controlling, in L^p, the quantity

$$\int_0^T \sum_{i=1}^d (X_i P_{T-t} f)^2 (Y_t) dt.$$

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Thanks to the previous result, we are let with the problem of controlling, in L^p, the quantity

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We can use the chain rule to easily check that

$$\sum_{i=1}^d (X_i P_{T-t} f)^2 (Y_t) \leq \left(\frac{1}{2} \sum_{i=1}^d X_i^2 + X_0 + \frac{\partial}{\partial t} \right) (P_{T-t} f)^2 (Y_t)$$

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From Itô's formula, the quantity

$$\left(\frac{1}{2}\sum_{i=1}^{d} X_i^2 + X_0 + \frac{\partial}{\partial t}\right)(P_{T-t}f)^2(Y_t)$$

is the bounded variation part of the sub-martingale $(P_{T-t}f)^2(Y_t)$.

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is the bounded variation part of the sub-martingale $(P_{T-t}f)^2(Y_t)$. From Lenglart-Lépingle-Pratelli inequality, we have therefore

$$\begin{aligned}
&\mathbb{E}\left(\left(\int_0^T \left(\frac{1}{2}\sum_{i=1}^{d} X_i^2 + X_0 + \frac{\partial}{\partial t}\right)(P_{T-t}f)^2(Y_t)dt\right)^{\frac{p}{2}}\right)\\
\le & p^{p/2}\mathbb{E}\left(\sup_{0\le t\le T}\left((P_{T-t}f)^2(Y_t)\right)^{p/2}\right)\\
\le & p^{p/2}\left(\frac{p}{p-2}\right)^{p/2}\mathbb{E}(f(Y_T)^p)\\
\le & p^{p/2}\left(\frac{p}{p-2}\right)^{p/2}\|f\|_p^p.
\end{aligned}$$

Conclusion

As a conclusion we proved:

Theorem

For any $1 < p < \infty$, there is a constant C_p depending only on p such that for every $f \in L^p_\mu(\mathbb{M})$,

$$\|\mathcal{S}_A^T f\| \leq C_p \|A\| \|f\|_p.$$

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$$\|\mathcal{S}_A^T f\| \leq C_p \|A\| \|f\|_p.$$

In particular this constant does not depend on T and thus for every $f \in L^p_\mu(\mathbb{M})$,

$$\|\mathcal{S}_A f\| \leq C_p \|A\| \|f\|_p.$$

Application I

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Theorem

Assume that $-M \leq V \leq -m$, then for every $f \in L^p_\mu(\mathbb{M})$,

$$\|\Psi(-L)f\|_p \leq \left(C_p \|a\|_\infty + M \frac{\Psi(m)}{m} \right) \|f\|_p.$$

Application II

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Theorem

For any constant coefficient matrix A,

$$\left\| \sum_{i,j=1}^{d} A_{ij} R_i R_j f \right\|_p \leq C_p \|A\| \|f\|_p.$$

An alternative argument for the main estimate

We have

$$\begin{aligned}
&\int_{\mathbb{M}} (\mathcal{S}_A^T f) g d\mu \\
=&\mathbb{E}\left(\sum_{i,j=1}^d \int_0^T A_{ij}(T-t)(X_j P_{T-t} f)(Y_t) dB_t^i \sum_{i=1}^d \int_0^T (X_i P_{T-t} g)(Y_t) dB_t^i \right) \\
\le& \left\| \sum_{i,j=1}^d \int_0^T A_{ij}(T-t)(X_j P_{T-t} f)(Y_t) dB_t^i \right\|_p \\
&\times \left\| \sum_{i=1}^d \int_0^T (X_i P_{T-t} g)(Y_t) dB_t^i \right\|_q
\end{aligned}$$

Burkholder's domination inequality

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Theorem (Bañuelos-Wang)

If Y is differentially subordinate to X, then

$$\|Y\|_p \leq (p^* - 1)\|X\|_p, \qquad 1 < p < \infty.$$

An alternative argument for the main estimate

Coming back to our problem, we get

$$\left\| \sum_{i,j=1}^{d} \int_0^T A_{ij}(T-t)(X_j P_{T-t} f)(Y_t) dB_t^i \right\|_p$$

$$\leq \|A\|(p^* - 1) \left\| \sum_{i=1}^{d} \int_0^T (X_i P_{T-t} f)(Y_t) dB_t^i \right\|_p$$

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and thus we have

$$\int_{\mathbb{M}} (\mathcal{S}_A^T f) g d\mu \leq \|A\|(p^*-1) \left\| \sum_{i=1}^{d} \int_0^T (X_i P_{T-t} f)(Y_t) dB_t^i \right\|_p \times \left\| \sum_{i=1}^{d} \int_0^T (X_i P_{T-t} g)(Y_t) dB_t^i \right\|_q$$

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Using finally the same type of arguments as before, we may prove

$$\left\| \sum_{i=1}^{d} \int_0^T (X_i P_{T-t} f)(Y_t) dB_t^i \right\|_p \leq \frac{2^{2-\frac{1}{p}} p^2}{p-1} \|f\|_p.$$

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and thus

$$\int_{\mathbb{M}} (\mathcal{S}_A^T f) g d\mu \leq 8 \|A\| (p^* - 1) \frac{p^4}{(p-1)^2} \|f\|_p \|g\|_q.$$