Taxi walks and the hard-core model on $\mathbb{Z}^2$

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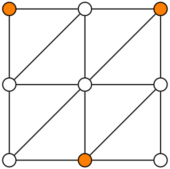
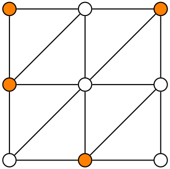
The hard-core model on a finite graph

A model of occupation of space by particles with non-negligible size



Valid configuration



Invalid configuration

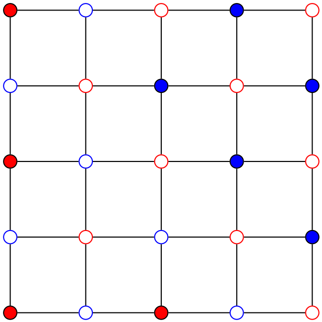
Density parameter $\lambda > 0$: Each valid configuration (*independent set*) I occurs with probability proportional to $\lambda^{|I|}$

A possible liquid-solid phase transition

- Small λ: typical configuration disordered
- Large λ: typical configuration mostly inside some maximum sized independent set

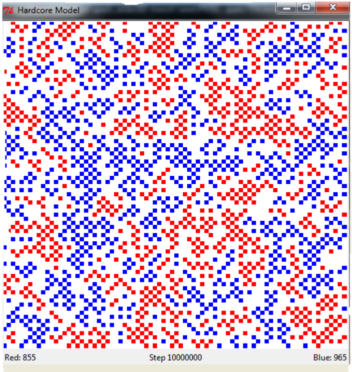
Example: boxes in $\mathbb{Z}^2$

$\mathbb{Z}^2$ has two maximum independent sets, the natural even and odd checkerboard sublattices (indicated by red and blue)

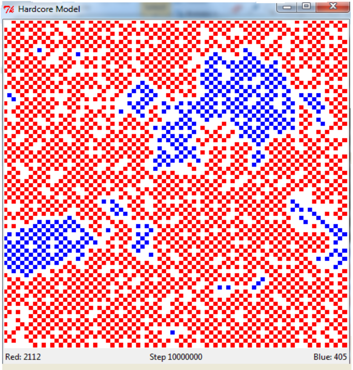


Simulations on a wrapped-around box in $\mathbb{Z}^2$

Some simulations (by Justin Hilyard)



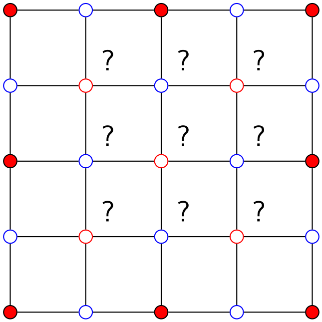
80×80, $\lambda = 2$



80×80, $\lambda = 5$

Conjecture: Model on boxes in $\mathbb{Z}^2$ flips from disorder to order around some λ_{crit}

Dealing with infinite graphs

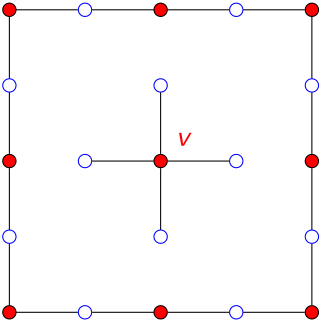


Gibbs measures à la Dobrushin, Lanford, Ruelle

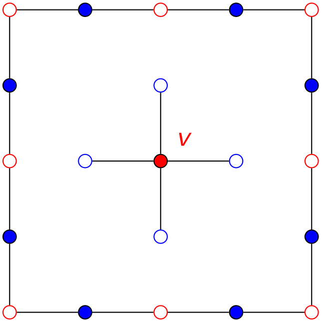
- Hardwire a boundary condition on a finite piece, and extend inside
- *Gibbs measure*: any limit measure as the finite pieces grow

Can different boundary conditions lead to different Gibbs measures?

The picture for large λ on $\mathbb{Z}^2$



$\mu^{\text{red}}(v \in I)$ likely



$\mu^{\text{blue}}(v \in I)$ unlikely

For large λ "influence of boundary" should persist

- $\mu^{\text{red}}(v \in I) > \mu^{\text{blue}}(v \in I)$ *equivalent* to multiple Gibbs measures
- $\mu^{\text{blue}}(v \in I)$ *small* forces $\mu^{\text{red}}(v \in I)$ *large*, so enough to show

$$\mu^{\text{blue}}(v \in I) \text{ small}$$

A precise conjecture

Conjecture (folklore, 1950's): There is $\lambda_{\text{crit}} \approx 3.796$ such that

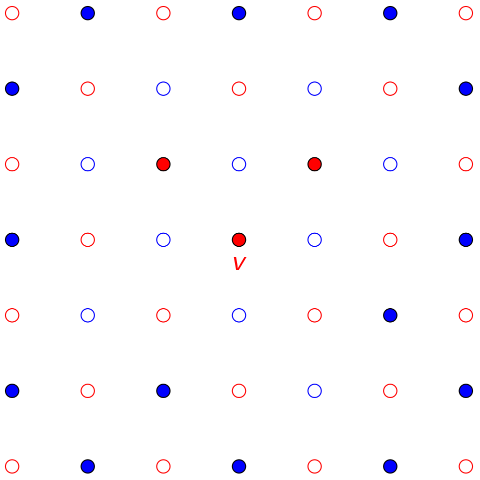
- for $\lambda < \lambda_{\text{crit}}$, hard-core model on $\mathbb{Z}^2$ has unique Gibbs measure
- for $\lambda > \lambda_{\text{crit}}$, there is phase coexistence (multiple Gibbs measures)

What's known (if λ_{crit} exists)

- Dobrushin (1968): $\lambda_{\text{crit}} > .25$
  (meaning: for $\lambda \leq .25$ there is unique Gibbs measure)
- Restrepo-Shin-Tetali-Vigoda-Yang (2011): $\lambda_{\text{crit}} > 2.38$
- Dobrushin (1968): $\lambda_{\text{crit}} < C$ for some large C
  (meaning: for $\lambda \geq C$ there are multiple Gibbs measures)
- Borgs-G. (2002-2011): $\lambda_{\text{crit}} < 300$, with 80 as theoretical limit

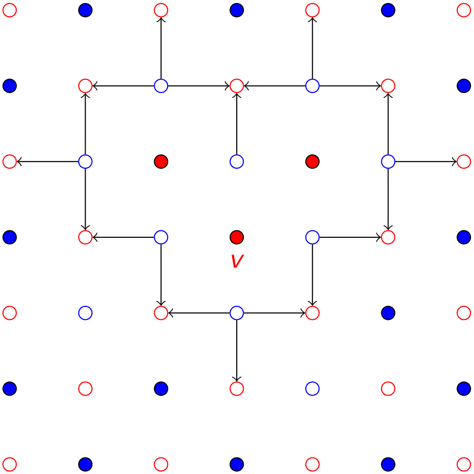
Theorem (Blanca-G.-Randall-Tetali 2012): $\lambda_{\text{crit}} < 5.3646$

The Peierls argument for phase coexistence



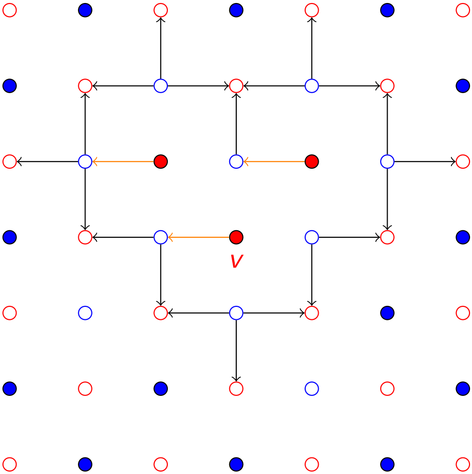
Blue boundary, red center ...

The Peierls argument for phase coexistence



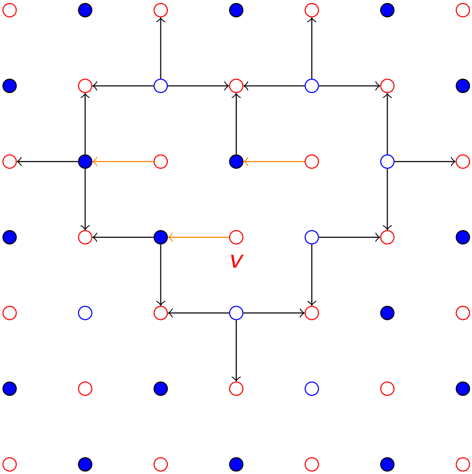
... leads to separating *contour* ...

The Peierls argument for phase coexistence



... shifting inside contour creates a more ordered independent set ...

The Peierls argument for phase coexistence



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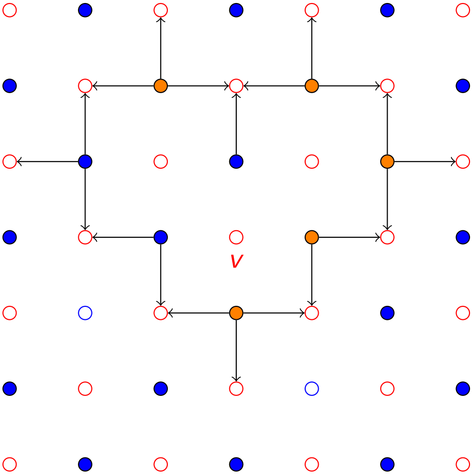
The Peierls argument for phase coexistence



... and frees up some vertices (in orange) that can be added

The Peierls argument for phase coexistence

Facts about contours

- Minimal unoccupied edge cutset separating v from boundary
- Interior-exterior edges always from blue sublattice to red
- Length 4ℓ for some $\ell \geq 3$, with ℓ edges in each direction
- Shift in any direction frees up ℓ vertices to be (potentially) added

Using contours

- One-to-many map with image weight $(1+\lambda)^\ell$ times larger than input
- Overlap of images controlled by number of possible contours
- The Peierls bound:

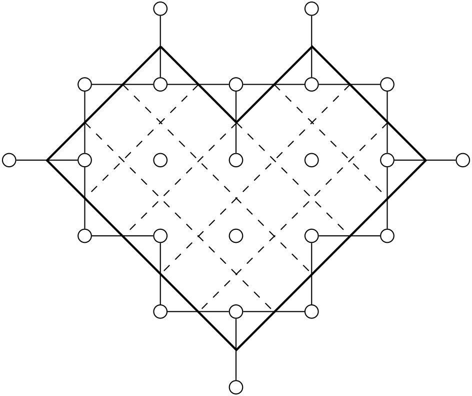
$$\mu^{\mathrm{blue}}(v \in I) \leq \sum_{\ell \geq 3} \frac{f_{\mathrm{contour}}(\ell)}{(1+\lambda)^\ell}$$

where $f_{\mathrm{contour}}(\ell)$ is number of contours of length 4ℓ

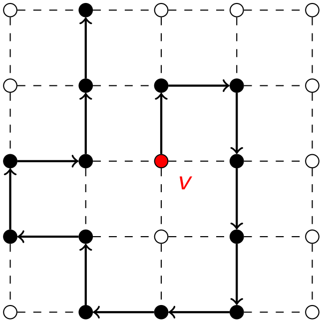
Contours are polygons

Contours are *simple polygons* in a rotated, dilated copy of $\mathbb{Z}^2$ where

- vertices the midpoints of edges of $\mathbb{Z}^2$
- vertices adjacent if their corresponding edges meet perpendicularly



Self-avoiding walks and an easy bound on λ_{crit}



A length 13 self-avoiding walk on $\mathbb{Z}^2$

- $\mathrm{SAW}(n) = \#(\text{walks of length } n) \leq 4 \times 3^{n-1}$
- $f_{\text{contour}}(\ell) \leq \mathrm{poly}(\ell)\mathrm{SAW}(4\ell - 1) \leq \mathrm{poly}(\ell)3^{4\ell} \approx 81^{\ell}$

Upper bounds on λ_{crit} using Peierls

An easy bound

- For $\lambda > 300$, $\mu^{\text{blue}}(v \in I) \leq \sum_{\ell \geq 3} \frac{f_{\text{contour}}(\ell)}{(1+\lambda)^{\ell}} < 1/10$
- Small enough for phase coexistence
- Theoretical limit: $\lambda_{\text{crit}} < 80 + \varepsilon$

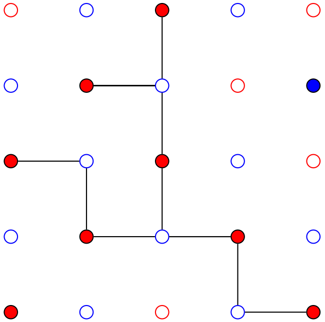
The connective constant μ_{SAW}

- $\text{SAW}(n+m) \leq \text{SAW}(n)\text{SAW}(m)$ (by concatenation)
- $\lim_{n\to\infty} \text{SAW}(n)^{\frac{1}{n}} = \inf_{n\to\infty} \text{SAW}(n)^{\frac{1}{n}} = \mu_{\text{SAW}}$ (by Fekete)
- $\text{SAW}(n) = \text{subexp}(n)\mu_{\text{SAW}}^{n}$

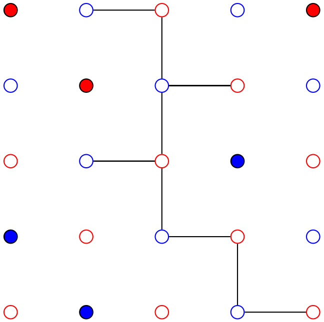
Better bounds

- Theoretical limit $\lambda_{\text{crit}} < \mu_{\text{SAW}}^{4} - 1 + \varepsilon$
- $\mu_{\text{SAW}} \approx 2.64$ gives $\lambda_{\text{crit}} \approx 48$
- Best rigorous bound: $\lambda_{\text{crit}} < 120$

Improving things – crosses and fault lines



Red cross



Fault line

Theorem (Randall 2006) Every independent set in a box has *one* of

- red cross
- blue cross
- fault line

Improving things – long contours

A new event that distinguishes between μ^{blue} and μ^{red}

- $E = \{I : I$ has red cross or fault line in m by m box$\}$
- In n by n box with blue boundary condition, I with red cross or fault line in smaller m by m box has contour of length $m/10$
- Peierls argument gives

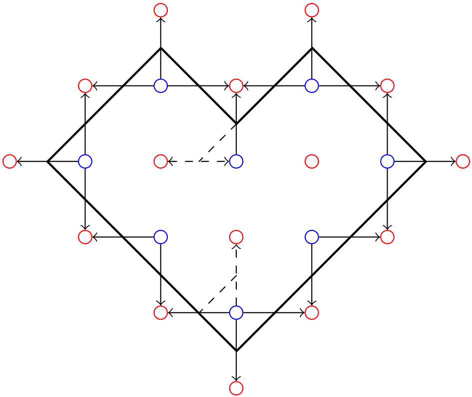
$$\mu^{\text{blue}}(E) \leq \sum_{\ell \geq m/40} \frac{f_{\text{contour}}(\ell)}{(1+\lambda)^\ell}$$

- $\mu^{\text{blue}}(E)$ small forces $\mu^{\text{red}}(E)$ large
- Large m absorbs $\text{subexp}(\ell)$ terms in estimates of $f_{\text{contour}}(\ell)$

Theorem: For all $\varepsilon > 0$,

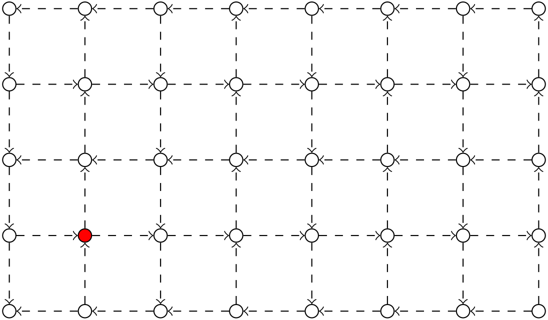
$$\lambda_{\text{crit}} < \mu_{\text{SAW}}^4 - 1 + \varepsilon$$

Improving things – contours have extra properties



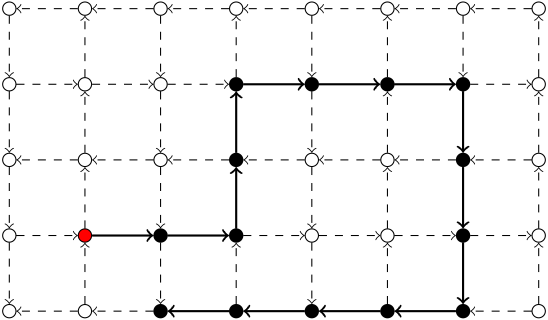
- Two consecutive turns not allowed
- Turn direction forced by parity of length of straight segments

Improving things – taxi walks



The Manhattan lattice

Improving things – taxi walks



A length 14 taxi walk on $\mathbb{Z}^2$

- Contours are closed taxi walks!
- $\lambda_{\text{crit}} < \mu_t^4 - 1 + \varepsilon$, where μ_t is taxi walk connective constant

Estimating μ_t

An easy bound

- Taxi walk encoded by $\{s,t\}$−string, no tt, so $TW(n) = O\left(1.618^n\right)$
- $\lambda_{\text{crit}} < 5.86$

Alm's method for fixed $m < n$

- γ_i, γ_j the ith and jth walks of length m
- a_{ij} is number of length n walks starting γ_i, ending γ_j
- $A = (a_{ij})$
- $\mu_t \leq \lambda_1(A)^{\frac{1}{n-m}}$

Taking $m = 20$, $n = 60$ (10057 by 10057 matrix), get

$$\mu_t < 1.59, \ \lambda_{\text{crit}} < 5.3646$$

Summary

New ideas

- Hard-core contours are more than just simple polygons
- Distinguishing events with long contours are worth hunting for!

Future work

- Improve upper bounds on μ_t
- Get *lower* bounds on μ_t (current limit for λ_{crit} is ≈ 4.22)
- Add new idea to explain 3.796
- Prove monotonicity – the *existence* of λ_{crit}

Future work?

THANK YOU!