Curvature-dimension inequalities for contact manifolds and gradient estimates for associated Markov semigroups

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Based on joint works with Fabrice Baudoin

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The Bakry-Émery criterion is typically used to establish functional inequalities for the invariant measure of L: Spectral gap inequality, Log-Sobolev inequality and hypercontractivity of the corresponding Markov semigroup.

Curvature dimension inequality

Typical examples of diffusion operators satisfying a curvature dimension inequality include the Laplace-Beltrami operators on Riemannian manifolds.

Curvature dimension inequality

Typical examples of diffusion operators satisfying a curvature dimension inequality include the Laplace-Beltrami operators on Riemannian manifolds. More precisely, from Bochner's formula

Theorem

The Laplace-Beltrami operator on Δ *on the Riemannian manifold* $\mathbb{M}$ *satisfies* $CD(\rho, d)$*, i.e.,*

$$\Gamma_2(f) \geq \frac{1}{d}(\Delta f)^2 + \rho\Gamma(f), \quad f \in C^\infty(\mathbb{M})$$

if and only if

$$\dim \mathbb{M} \leq d, \quad \mathbf{Ric} \geq \rho.$$

Gradient bounds

There is a Markov semigroup $P_t = e^{tL}$ associated with L. In probabilistic sense, there is a unique continuous Markov process X_t associated with P_t such that

$$P_t f(x) = \mathbb{E}(f(X_t)|X_0 = x).$$

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Theorem

We have that

$$\Gamma_2(f) \geq \rho\Gamma(f), \quad \forall f \in C^\infty(\mathbb{M})$$

if and only if
$$\Gamma(P_t(f)) \leq e^{-2\rho t} P_t\Gamma(f), \quad \forall f \in C^\infty(\mathbb{M}).$$

Associated Markov process

Sketch of the proof.

Consider the function $\phi(s) = P_s\Gamma(P_{t-s}f)$, by easy calculation, we have that

$$\phi'(s) = 2P_s\Gamma_2(P_{t-s}f),$$

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hence we obtain:

$$\Gamma_2(f) \geq \rho\Gamma(f) \Rightarrow \phi'(t) \geq 2\rho\phi(t) \Rightarrow \Gamma(P_t(f)) \leq e^{-2\rho t}P_t\Gamma(f).$$

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On the other hand, by taking small-time asymptotic, we have that:

$$\Gamma(P_t(f)) \leq e^{-2\rho t}P_t\Gamma(f) \Rightarrow$$

$$\Gamma(f) + 2\Gamma(f, Lf)t \leq \Gamma(f) + (L\Gamma(f) - 2\rho\Gamma(f))\, t$$

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μ is an invariant measure of this Markov process, i.e., for any positive function $f \in L^1(\mu)$,

$$\int_{\mathbb{M}} P_t f(x)\mu(dx) = \int_{\mathbb{M}} f(x)\mu(dx).$$

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Questions:

- Is μ a finite measure (probability measure)?
- Does the heat semigroup $P_t(f)$ converge to the equilibrium when $t \to \infty$? i.e.,

$$P_t f(x) \xrightarrow{?}_{t\to\infty} \frac{1}{\mu(\mathbb{M})} \int_{\mathbb{M}} f d\mu.$$

Curvature dimension inequality

Theorem

If $\Gamma_2 \geq \rho\Gamma$ with $\rho > 0$, then the measure μ is finite and for every $x \in \mathbb{M}$, $f \in L^2(\mathbb{M})$,

$$P_t f(x) \longrightarrow_{t\to\infty} \frac{1}{\mu(\mathbb{M})} \int_{\mathbb{M}} f d\mu.$$

In other words,

$$X_t \xrightarrow{\mathcal{D}}_{t\to+\infty} \frac{1}{\mu(\mathbb{M})}\mu$$

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$$\implies \mu(\mathbb{M}) < \infty$$

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Subelliptic diffusion operators

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requires some form of ellipticity of the generator L(invertibility of the diffusion matrix). Such criterion fails to hold even for simple subelliptic diffusion operators, like the sub-Laplacian on the Heisenberg group (Juillet).

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Subelliptic diffusion operators

Let L be a diffusion operator defined on a manifold $\mathbb{M}$. We assume that L is symmetric with respect to a smooth measure μ and (locally) subelliptic. Elliptic operators are subelliptic .For subelliptic operators, the corresponding Markov has a smooth density. Another important subclass is given by sum of squares type operators

$$L = -\sum_{i=1}^{d} X_i^* X_i$$

that satisfy the Hörmander's bracket generating condition.

Vertical form

We will assume, additionally, that $\mathbb{M}$ is endowed with a first-order differential bilinear form $\Gamma^T(f,g)$.

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Γ^T may be thought as a vertical complement to Γ. We can then define the *intrinsic* curvature of L with respect to Γ^T.

$$\Gamma_2^T(f,g) = \frac{1}{2}\left(L\Gamma^T(f,g) - \Gamma^T(f,Lg) - \Gamma^T(Lf,g)\right).$$

Example: Contact Riemannian manifolds

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In this example, the vertical direction is canonical and given by the Reeb vector field T:

$$\Gamma^T(f) = (Tf)^2.$$

Example: Contact Riemannian manifolds with Sasakian structure

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Theorem (Baudoin-Garofalo)

Let $\mathbb{M}$ be a contact CR Sasakian manifold. We have $\mathbf{Ric}_{\nabla} \geq \rho_1$ *if and only if*

$$\Gamma_2(f) + 2\sqrt{\Gamma(f)\Gamma_2^T(f)} \geq \frac{1}{2n}(Lf)^2 + \rho_1\Gamma(f) + \frac{n}{2}\Gamma^T(f).$$

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It is equivalent to the fact that for every $\nu > 0$,

$$\Gamma_2(f) + \nu\Gamma_2^T(f) \geq \frac{1}{2n}(Lf)^2 + \left(\rho_1 - \frac{1}{\nu}\right)\Gamma(f) + \frac{n}{2}\Gamma^T(f).$$

Example: 3-dimensional model spaces

For $\rho \in \mathbb{R}$, $\mathbb{G}(\rho)$ is a n-dimensional Lie group whose Lie algebra $\mathfrak{g}$ has a basis $\{X, Y, Z\}$ such that

$$[X, Y] = Z, \quad [Z, X] = -\rho Y, \quad [Y, Z] = \rho X$$

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Moreover,

$$\Gamma_2(f) = ||\nabla^2_{\mathcal{H}} f||^2 + \rho\Gamma(f) + \frac{1}{2}\Gamma^T(f) + 2(Yf(XZf) - Xf(YZf)),$$

$$\Gamma_2^T(f) = (XZf)^2 f + (YZf)^2.$$

where

$$\nabla^2_H f = \begin{pmatrix} X^2 f & \frac{1}{2}(XYf + YXf) \\ \frac{1}{2}(XYf + YXf) & Y^2 f \end{pmatrix}$$

Example: 3-dimensional model spaces

Hence we can obtain the generalized curvature dimension inequality: for all $f \in C^\infty(\mathbb{G}(\rho))$ and any $\nu > 0$,

$$\Gamma^2(f) + \nu\Gamma_2^T(f) \geq \frac{1}{2}(Lf)^2 + \left(\rho - \frac{1}{\nu}\right)\Gamma(f) + \frac{1}{2}\Gamma^T(f).$$

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- $\rho = 0$: The "flat" Heisenberg group;
- $\rho = 1$: The Lie group $SU(2)$;
- $\rho = -1$:The Lie group $\widetilde{SL(2)}$.

Generalized curvature dimension inequality

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Definition

We say that L satisfies the generalized-curvature dimension inequality $CD(\rho_1, \rho_2, \kappa, d)$ if for every $\nu > 0$,

$$\Gamma_2(f,f) + \nu \Gamma_2^T(f,f) \geq \frac{1}{d}(Lf)^2 + \left(\rho_1 - \frac{\kappa}{\nu}\right)\Gamma(f,f) + \rho_2 \Gamma^T(f,f).$$

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We say a contact manifold satisfying the Sasakian condition if the first-order differential bilinear form $\Gamma^T(f,g)$ satisfies

$$\Gamma(f, \Gamma^T(f)) = \Gamma^T(f, \Gamma(f)).$$

Gradient estimates for the associated heat semigroup

Theorem (F. Baudoin, M. Bonnefont)

Let $f \in L^2(\mathbb{M})$ *such that* $f \in C^\infty(\mathbb{M})$ *and* $\Gamma(f,f)$, $\Gamma^T(f,f) \in L^1(\mathbb{M})$. *For* $x \in \mathbb{M}$, $t \geq 0$, *when* $\rho_1 \geq 0$ *one has*

$$\Gamma(P_t f) + \frac{\kappa + \rho_2}{\rho_1} \Gamma^T(P_t f) \leq e^{\frac{-2\rho_1\rho_2}{\kappa+\rho_2} t} \left(P_t(\Gamma(f)) + \frac{\kappa + \rho_2}{\rho_1} P_t(\Gamma^T(f)) \right)$$

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Corollary

The measure μ *is finite, i.e.,* $\mu(\mathbb{M}) < +\infty$ *and for every* $x \in \mathbb{M}$, $f \in L^2(\mathbb{M})$,

$$P_t f(x) \to_{t\to\infty} \frac{1}{\mu(\mathbb{M})} \int_{\mathbb{M}} f d\mu.$$

Spectral gap inequality

Theorem

Assume that L satisfies the generalized curvature inequality $CD(\rho_1, \rho_2, \kappa)$ with $\rho_1 > 0$, $\rho_2 > 0$ and $\kappa \geq 0$. The measure μ is finite and the following Poincaré inequality holds:

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We may rewrite the inequality in the form

$$\sigma^2(f) \leq \frac{\kappa + \rho_2}{\rho_1 \rho_2} \mathcal{E}(f, f),$$

where $\sigma^2(f)$ denotes the variance of the function f with respect to μ, and $\mathcal{E}(f, f) = \int \Gamma(f) d\mu$ stands for the energy of f.

Subelliptic Myers theorem

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If the inequality $CD(\rho_1, \rho_2, \kappa, n)$ *holds for some constants* $\rho_1 > 0, \rho_2 > 0, \kappa > 0$, *then the metric space* $(\mathbb{M}, d)$ *is compact*

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Theorem

If $\mathbb{M}$ *is a Sasakian manifold such that* $\mathbf{Ric}_\nabla \ge \rho_1$, *with* $\rho_1 > 0$, *then the metric space* $(\mathbb{M}, d)$ *is compact and we have*

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Generalized CD inequality on contact manifolds

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Theorem (Baudoin-J. Wang)

Under suitable geometric conditions,

$$\Gamma_2(f) + \nu \Gamma_2^Z(f) \geq \left(\rho_1 - \frac{\kappa_1}{\nu}\right) \Gamma(f) + \left(\rho_2 - \kappa_2 \nu^2\right) \Gamma^Z(f)$$

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$\mathbb{M}$ is a compact Riemannian manifold with contact structure. Let $f \in L^2(\mathbb{M})$ such that $f \in C^\infty(\mathbb{M})$ and $\Gamma(f,f)$, $\Gamma^T(f,f) \in L^1(\mathbb{M})$. For $x \in \mathbb{M}$, $t > 0$, when $\rho_1 - \frac{\kappa\sqrt{\rho_3}}{\sqrt{\rho_2}} \geq 0$ we have

$$\Gamma(P_t f) + C_{\rho_1,\rho_2,\kappa,\kappa_2}\Gamma^T(P_t f) \leq e^{-\alpha t}\left(P_t(\Gamma(f)) + C_{\rho_1,\rho_2,\kappa,\kappa_2}P_t(\Gamma^T(f))\right)$$

where $C_{\rho_1,\rho_2,\kappa,\kappa_2} = \frac{\alpha+\sqrt{\alpha^2+16\rho_2\kappa_2}}{4\rho_2}$ and $\alpha = \frac{2\rho_1\rho_2-2\kappa\sqrt{\rho_2\kappa_2}}{(\rho_2+\kappa)}$.

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Corollary

For every $x \in \mathbb{M}$, $f \in L^2(\mathbb{M})$,

$$P_t f(x) \to_{t\to\infty} \frac{1}{\mu(\mathbb{M})}\int_{\mathbb{M}} f d\mu.$$

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In other words,

$$\sigma^2(f) \le \frac{-2\rho_1\rho_2 + 2\kappa\sqrt{\rho_2\kappa_2}}{(\rho_2+\kappa)} \mathcal{E}(f,f),$$

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