Short-time asymptotics for ATM option prices under tempered stable processes

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Outline

Lévy Process

1. Lévy process $\{X_t\}_{t\geq 0}$
   - $X_0 = 0$
   - Independent Increments:

$$t_0 < \cdots < t_n \implies X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}} \text{ are independent}$$

   - Stationary Increments

$$s < t \implies X_t - X_s \stackrel{\mathfrak{D}}{=} X_{t-s}$$

   - Paths $t \to X_t(\omega)$ that are right-continuous with left-limits
   - $X_s \xrightarrow{\mathbb{P}} X_t \quad \text{when} \quad s \to t$
2. The distribution law of $\{X_t\}_{t\geq 0}$ is determined by the distribution of X_1:
   - If $\mathcal{L}(X_1) \sim \mathcal{N}(0, 1)$, then $X_t = W_t$ is the standard Brownian Motion;
   - If $\mathcal{L}(X_1) \sim \mathit{Poisson}(\lambda)$, then $X_t = N_t$ is a Poisson process with intensity λ;

Tempered Stable Processes (Rosiński, 2007)

1. Let $\alpha \in (0,2)$, $b \in \mathbb{R}$, and $q^+, q^- : (0,\infty) \to [0,\infty)$ completely monotone functions with $q^\pm(\infty) = 0$ and $q^\pm(0^+) < \infty$:

$$(-1)^k \frac{d^k q^\pm}{dx^k}(x) \geq 0, \qquad (k = 0, 1, \dots).$$

2. A Tempered Stable Process ($T\alpha S$) is a Lévy process $\{X_t\}_{t\geq 0}$ whose distribution at $t = 1$ has the characteristic function:

$$\mathbb{E}\left(e^{iuX_1}\right) = \exp\left(ibu + \int_{\mathbb{R}\setminus\{0\}} \left(e^{iux} - 1 - iux\mathbf{1}_{\{|x|\leq 1\}}\right) s(x)dx\right),$$

with

$$s(x) = |x|^{-\alpha-1} q^+(x)\mathbf{1}_{x>0} + |x|^{-\alpha-1} q^-(-x)\mathbf{1}_{x<0}.$$

3. b, α, and s are called the “drift", index, and Lévy density of the $T\alpha S$ process.

Connection to Stable Processes

1. If q^+, q^- are constants, then the resulting Lévy process is a Stable Lévy Process $\{Z_t\}_{t\geq 0}$;
2. For a suitable $c \in \mathbb{R}$, the drifted process $\bar{Z}_t := Z_t - ct$ is self-similar:

$$\{h^{-1/\alpha}\bar{Z}_{ht}\}_{t\geq 0} \stackrel{\mathfrak{D}}{=} \{\bar{Z}_t\}_{t\geq 0} \qquad (h > 0).$$

   If $c = 0$, we say Z is strictly α-stable
3. Distributions are too "fat" for applications:

$$\mathbb{E}(|Z_t|^p) = \infty, \qquad \text{for any} \qquad p > \alpha.$$

4. Being strictly decreasing, the functions q^+, q^- "temper" the jump intensity of the associate stable process:

$$\mathbb{E}(|X_t|^p) < \infty \iff \int_{|x|\geq 1} |x|^{p-\alpha-1} q(x)dx < \infty.$$

Short and long time behavior

1. In short-time or locally, $\{X_t\}_{t\geq 0}$ behaves like a stable process:
   - $1 < \alpha < 2$:
   $$\{h^{-1/\alpha} X_{ht}\} \xrightarrow{\mathfrak{D}} \{Z_t\}_{t\geq 0}, \qquad (h \to 0),$$
   for a strictly α-stable process $\{Z_t\}_{t\geq 0}$;
   - $0 < \alpha < 1$:
   $$\{h^{-1/\alpha} (X_{ht} - cht)\} \xrightarrow{\mathfrak{D}} \{Z_t\}_{t\geq 0}, \qquad (h \to 0),$$
   for a suitable drift c and strictly α-stable process $\{Z_t\}_{t\geq 0}$;
2. In long-time, $\{X_t\}_{t\geq 0}$ behaves like a Brownian Motion:
$$\{h^{-1/2} X_{ht}\} \xrightarrow{\mathfrak{D}} \{B_t\}_{t\geq 0}, \qquad (h \to \infty),$$

where $\{B_t\}_{t\geq 0}$ is a suitable Brownian motion.

The problem

1. Consider a $T\alpha S$ Process $\{X_t\}_{t\geq 0}$ with finite exponential moment:

$$\mathbb{E}\left(e^{X_t}\right) < \infty \quad \Longleftrightarrow \quad \int_1^\infty e^x x^{-\alpha-1} q(x) dx < \infty.$$

2. The "drift" b of X is such that $S_t := e^{X_t}$ is a $\mathbb{P}$-martingale:

$$u < v : \quad \mathbb{E}\left(S_v \middle| S_t, t \leq u\right) = S_u \quad \Longleftrightarrow \quad \mathbb{E}\left(e^{X_1}\right) = 1.$$

3. Consider the functional:

$$\Pi_t := \mathbb{E}\left[\left(e^{X_t} - 1\right)^+\right] = \mathbb{E}\left[\left(e^{X_t} - 1\right) \mathbf{1}_{X_t \geq 0}\right].$$

4. By DCT, $\Pi_t \to 0$ when $t \to 0$.

General Problem:

We want to determine the rate of convergence as $t \to 0$.

Motivation

1. In mathematical finance,

$$\Pi_t = \mathbb{E}\left[\,\left(e^{X_t} - 1\right)^+\,\right]$$

is interpreted as the price of an ATM European call option with expiry t at time 0 written on a stock whose price process is modeled by $S_t := e^{X_t}$.

2. Our results shed light on the behavior of option prices close to expiration under an exponential Lévy model.

3. The European call option price with expiry t and strike $K = e^{\kappa}$ is

$$\Pi_t(K) = \mathbb{E}\left[(S_t - K)^+\right] = \mathbb{E}\left[\,\left(e^{X_t} - e^{\kappa}\right)^+\,\right].$$

4. In mathematics, $\varphi_K(S) = (S - K)^+$ are natural building blocks of convex functions $f : \mathbb{R}_+ \to \mathbb{R}_+$:

$$f(S) = f(0) + f'_+(0)S + \int_0^\infty (S - K)^+ \mu(dK).$$

Some relevant literature

Two distinct regimes: Not ATM and ATM.

- Not ATM ($\kappa \neq 0$)
  1. Tankov (2011): Leading order term for general Lévy process:

$$\mathbb{E}\left(e^{X_t} - e^{\kappa}\right)^{+} = (1 - e^{\kappa})^{+} + t \int \left(e^{x} - e^{\kappa}\right)^{+} s(x)dx + o(t).$$

  2. F-L & Forde (2012): High-order term for relatively general Lévy process;

$$\mathbb{E}\left(e^{X_t} - e^{\kappa}\right)^{+} = (1 - e^{\kappa})^{+} + t \int \left(e^{x} - e^{\kappa}\right)^{+} s(x)dx + \frac{t^2}{2} d_2(\kappa) + o(t^2),$$

  where $d_2(\kappa)$ has an explicit form in terms of s.

Some relevant literature. Cont...

- ATM ($\kappa = 0$)
  1. Roper (2011), Tankov (2011): Leading order term for bounded variation process ($\alpha < 1$):
  $$\mathbb{E}\left(e^{X_t} - 1\right)^+ = t \max\left\{\int \left(e^x - 1\right)^+ s(x)dx, \int \left(1 - e^x\right)^+ s(x)dx\right\} + o(t).$$
  2. Tankov (2011), F-L & Forde (2012), Muhle-Karbe & Nutz (2011): Leading term for Lévy process with stable-like small-time behavior with $\alpha > 1$:
  $$\mathbb{E}\left(e^{X_t} - 1\right)^+ = t^{1/\alpha}\,\mathbb{E}\left(Z_1^+\right) + o\left(t^{1/\alpha}\right), \qquad (t \to 0)$$
  where $\{Z_t\}_{t\geq 0}$ is a centered α-stable process.
  3. Intuition: In light of the Taylor expansion of exponential,
  $$t^{-1/\alpha}\mathbb{E}\left(e^{X_t} - 1\right)^+ \approx t^{-1/\alpha}\mathbb{E}\left(X_t\right)^+ = \mathbb{E}\left(t^{-1/\alpha}X_t\right)^+ \xrightarrow{t\to 0} \mathbb{E}\left(Z_1^+\right).$$
  4. If s is symmetric ($q^+(x) = q^-(x)$), then
  $$d_1 := \mathbb{E}(Z_1^+) = \frac{1}{\pi}\Gamma\left(1 - 1/\alpha\right)\left(2q^{\pm}(0)\Gamma(-\alpha)\left|\cos\left(\pi\alpha/2\right)\right|\right)^{1/\alpha}.$$

Assumptions and Notation

The index α is in $(1,2)$ and the function $q:\mathbb{R}\backslash\{0\}\to[0,\infty)$ in $s(x):=q(x)|x|^{-\alpha-1}$ is such that

$$\text{(i) } C_+ := \lim_{x\searrow 0} q(x) < \infty, \qquad \text{(ii) } C_- := \lim_{x\nearrow 0} q(x) < \infty$$

$$\text{(iii) } \lim_{x\searrow 0} -q'(x) < \infty, \qquad \text{(iv) } \lim_{x\nearrow 0} q'(x) < \infty,$$

$$\text{(iv) } \int_1^\infty e^x x^{-\alpha-1} q(x)dx < \infty, \qquad \text{(v) } \limsup_{|x|\to\infty} \frac{|\ln q(x)|}{|x|} < \infty.$$

We define the standardized function $\bar{q}$:

$$\bar{q}(x) := \frac{q(x)}{C_+}\mathbf{1}_{x>0} + \frac{q(x)}{C_-}\mathbf{1}_{x<0},$$

so that $\bar{q}(0) := \lim_{|x|\to 0}\bar{q}(x) = 1$ and $\bar{q}(x) \le 1$.

Main result 1

Under the exponential tempered stable model with $\alpha > 1$,

$$\Pi_t = \mathbb{E}\left(e^{X_t} - 1\right)^+ = d_1\, t^{\frac{1}{\alpha}} + d_2\, t + o(t), \qquad (t \to 0), \tag{1}$$

where $d_1 = \mathbb{E}(Z_1^+)$ and $d_2 = \vartheta + \gamma \mathbb{P}(Z_1 \geq 0)$ with

$$\vartheta := C_+ \int_0^\infty \left(e^x \bar{q}(x) - \bar{q}(x) - x\right) x^{-\alpha-1} dx$$

$$\gamma := b + \frac{C_+ + C_-}{\alpha - 1} + C_+ \int_0^\infty x^{-\alpha}\left(1 - \bar{q}(x)\right) dx + C_- \int_{-\infty}^0 |x|^{-\alpha}\left(1 - \bar{q}(x)\right) dx.$$

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Main result 2

Under the exponential $T\alpha S$-process $\{X_t\}_{t\geq 0}$ with an independent Brownian component $\{\sigma W_t\}_{t\geq 0}$,

$$\Pi_t = \mathbb{E}\left(e^{X_t+\sigma W_t} - 1\right)^+ = d_1\, t^{\frac{1}{2}} + d_2\, t^{\frac{3-\alpha}{2}} + o(t^{\frac{3-\alpha}{2}}), \qquad (t \to 0), \qquad (2)$$

where

$$d_1 := \sigma \mathbb{E}\left(W_1^+\right) = \frac{\sigma}{\sqrt{2\pi}}$$

$$d_2 := \frac{C_+ + C_-}{2\alpha(\alpha-1)}\sigma^{1-\alpha}\,\mathbb{E}\left(|W_1|^{1-\alpha}\right)$$

$$= \frac{2^{1-\alpha}}{\sqrt{\pi}}\Gamma\left(1-\frac{\alpha}{2}\right)\frac{(C_+ + C_-)\,\sigma^{1-\alpha}}{2\alpha(\alpha-1)}.$$

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where

$$\begin{aligned} d_1 &:= \sigma \mathbb{E}\left(W_1^+\right) = \frac{\sigma}{\sqrt{2\pi}} \\ d_2 &:= \frac{C_+ + C_-}{2\alpha(\alpha-1)}\, \sigma^{1-\alpha}\, \mathbb{E}\left(|W_1|^{1-\alpha}\right) \\ &= \frac{2^{1-\alpha}}{\sqrt{\pi}} \Gamma\left(1 - \frac{\alpha}{2}\right) \frac{(C_+ + C_-)\,\sigma^{1-\alpha}}{2\alpha(\alpha-1)}. \end{aligned}$$

CGMY Model $s(x) = C|x|^{-Y-1}\left(e^{-|x|/G}\mathbf{1}_{x<0} + e^{-|x|/M}\mathbf{1}_{x>0}\right)$

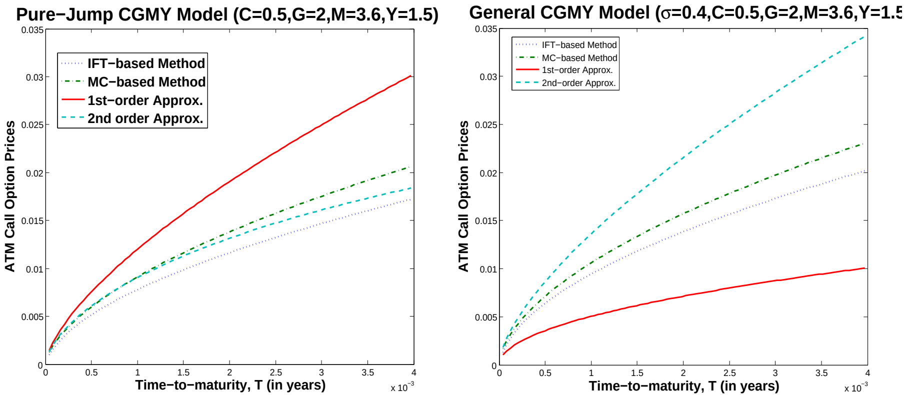

Figure: Comparisons of ATM call option prices computed by two methods (Inverse Fourier Transform and Monte-Carlo method) with the first- and second-order approximations.

Conclusions

1. Obtained the second-order short-time expansions for ATM European call option prices under a tempered stable process with a possible nonzero independent Brownian component.
2. Characterized explicitly the effects of the different parameters into the behavior of ATM option prices near expiration.
3. Introduce a new method of proof which can potentially be applied to “any" Lévy process having the fundamental property of being stable under a suitable change of probability measure and whose Lévy density can be “closely" approximated by a stable density near the origin.

For Further Reading I

Figueroa-López, Gong, & Houdré.
High-order short-time expansions for ATM option prices under a tempered stable Lévy model.
ArXiv, 2012.