Large Deviations and Slowdown Asymptotics of Excited Random Walks

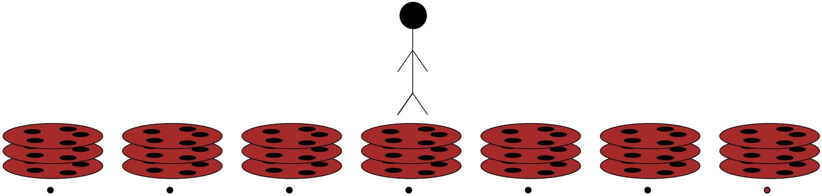
Jonathon Peterson

Department of Mathematics
Purdue University

September 4, 2012

Excited (Cookie) Random Walks

(M, p) **Cookie Random Walk**
Initially M cookies at each site.



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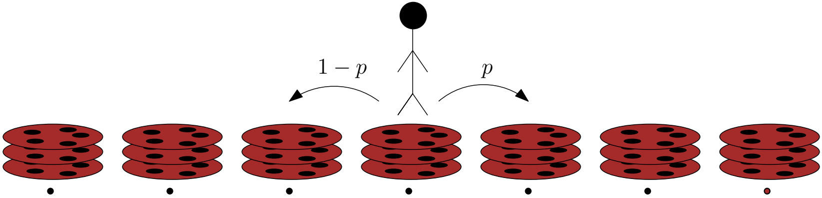

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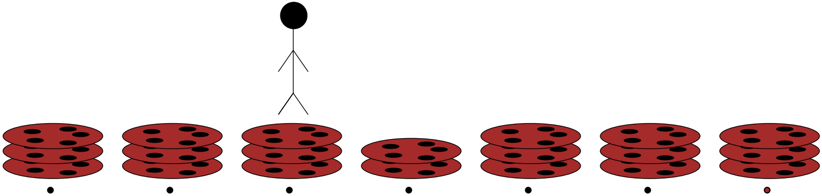

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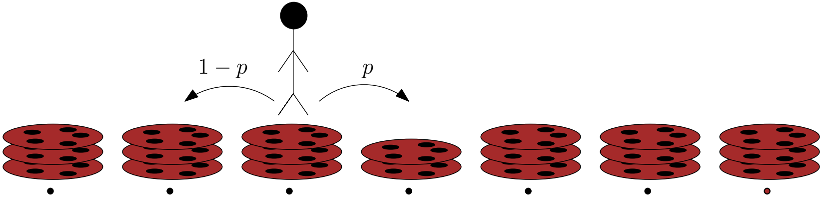

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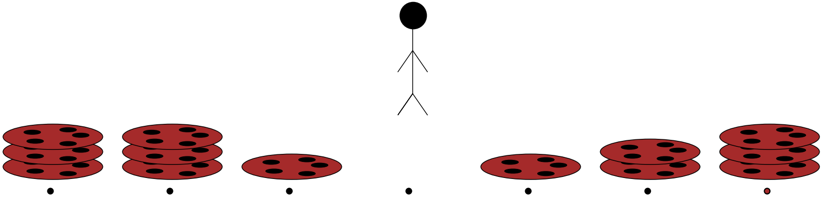

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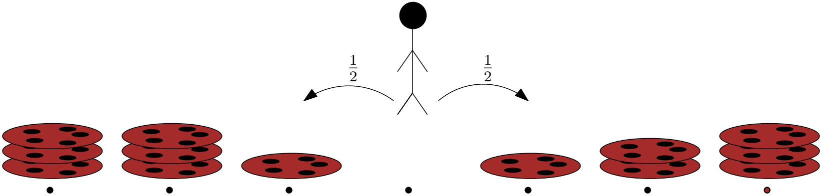

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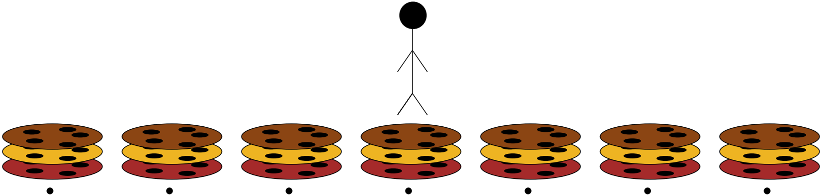

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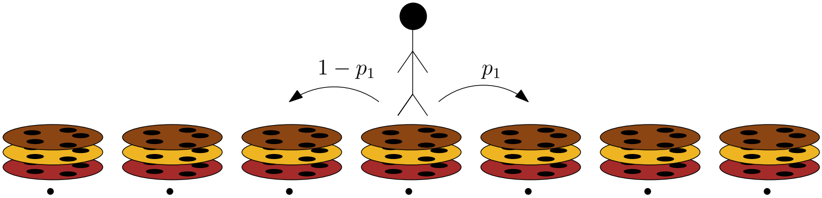

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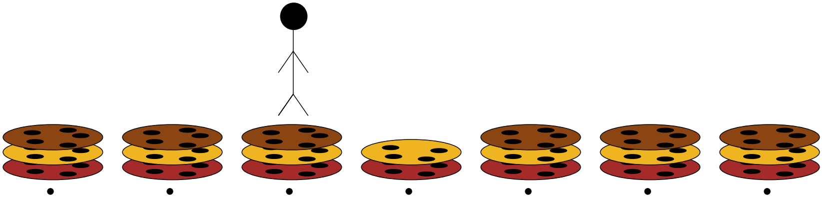

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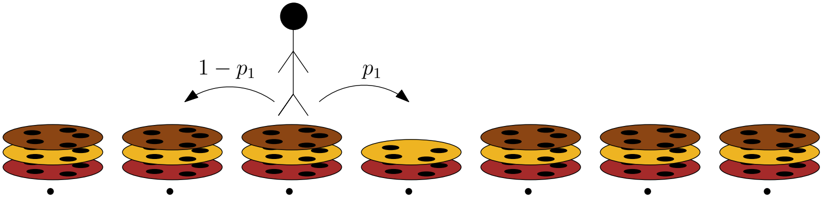

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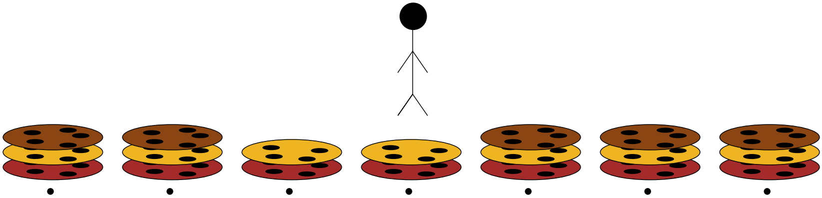

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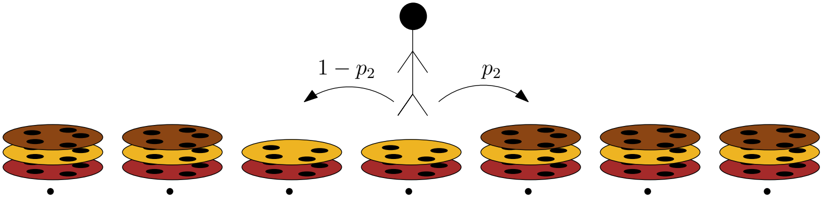
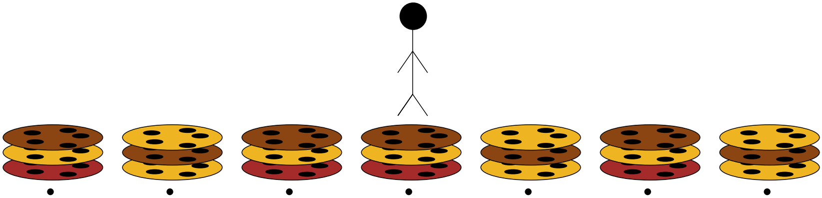


Excited (Cookie) Random Walks

Random i.i.d. cookie environments

- M cookies per site.
- $\omega_x(j)$ – strength of j-th cookie at site x.
- Cookie environment $\omega = \{\omega_x\}$ is i.i.d.
  Cookies *within* a stack may be dependent.



Recurrence/Transience and LLN

Average drift per site

$$\delta = E\left[\sum_{j=1}^{M}(2\omega_0(j) - 1)\right]$$

Theorem (Zerner '05, Zerner & Kosygina '08)

The cookie RW is recurrent if and only if $\delta \in [-1, 1]$.

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Theorem (Basdevant & Singh '07, Zerner & Kosygina '08)

$\lim_{n\to\infty} X_n/n = v_0$, *and* $v_0 > 0 \iff \delta > 2$.

No explicit formula is known for v_0.

Limiting Distributions for Excited Random Walks

Theorem (Basdevant & Singh '08, Kosygina & Zerner '08, Dolgopyat '11)

Excited random walks have the following limiting distributions.

Regime	**Re-scaling**	**Limiting Distribution**
$\delta \in (1,2)$	$\frac{X_n}{n^{\delta/2}}$	$\left(\frac{\delta}{2}\text{-stable}\right)^{-\delta/2}$
$\delta \in (2,4)$	$\frac{X_n - nv_0}{n^{2/\delta}}$	*Totally asymmetric* $\frac{\delta}{2}$*-stable*
$\delta > 4$	$\frac{X_n - nv_0}{A\sqrt{n}}$	*Gaussian*

Results are also known for other values of δ.

Note: $\delta > 1$ results similar to transient RWRE.

Limiting Distributions for Excited Random Walks

Theorem (Basdevant & Singh '08, Kosygina & Zerner '08, Dolgopyat '11)

Hitting times $T_n = \min\{k \geq 0 : X_k = n\}$ of excited random walks have the following limiting distributions.

Regime	***Re-scaling***	***Limiting Distribution***
$\delta \in (1, 2)$	$\frac{T_n}{n^{2/\delta}}$	*Totally asymmetric $\frac{\delta}{2}$-stable*
$\delta \in (2, 4)$	$\frac{T_n - n/v_0}{n^{2/\delta}}$	*Totally asymmetric $\frac{\delta}{2}$-stable*
$\delta > 4$	$\frac{T_n - n/v_0}{A\sqrt{n}}$	*Gaussian*

Results are also known for other values of δ.

Note: $\delta > 1$ results similar to transient RWRE.

Large Deviations for Excited Random Walks

Theorem (P. '12)

X_n/n has a large deviation principle with rate function $I_X(x)$. That is, for any open $G \subset [-1,1]$

$$\liminf_{n\to\infty} \frac{1}{n} \log P(X_n/n \in G) \geq -\inf_{x\in G} I_X(x)$$

and for any closed $F \subset [-1,1]$

$$\limsup_{n\to\infty} \frac{1}{n} \log P(X_n/n \in F) \leq -\inf_{x\in F} I_X(x)$$

Informally, $P(X_n \approx xn) \approx e^{-nI_X(x)}$.

Large Deviations for Hitting Times of Excited Random Walks

$$T_x = \inf\{n \geq 0 : X_n = x\}, \qquad x \in \mathbb{Z}.$$

Theorem (P. '12)

T_n/n has a large deviation principle with rate function $I_T(t)$.

Large Deviations for Hitting Times of Excited Random Walks

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T_n/n has a large deviation principle with rate function $I_T(t)$.
T_{-n}/n has a large deviation principle with rate function $\bar{I}_T(t)$.

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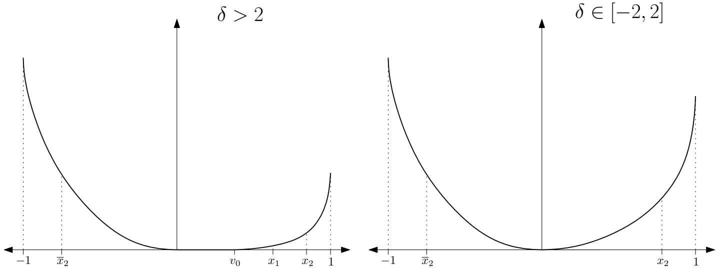
Implies LDP for X_n/n.

$$P(X_n > xn) \approx P(T_{xn} < n).$$

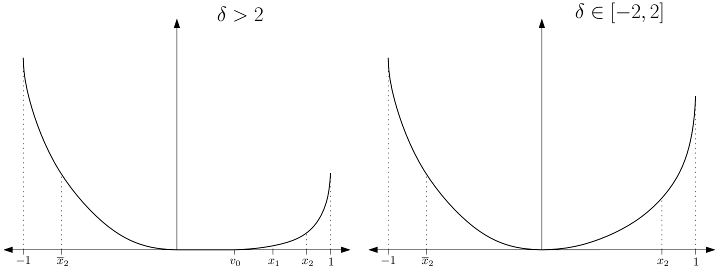
$$I_X(x) = \begin{cases} x I_T(1/x) & x \in (0,1] \\ 0 & x = 0 \\ |x| \bar{I}_T(1/|x|) & x \in [-1,0) \end{cases}$$

Properties of the rate function $I_X(x)$



1. $I_X(x)$ is a convex function.

Properties of the rate function $I_X(x)$



1. $I_X(x)$ is a convex function.
2. Zero Set
   - $\delta \in [-2, 2]$: $I_X(x) = 0 \iff x = 0$.
   - $\delta > 2$: $I_X(x) = 0 \iff x \in [0, v_0]$.

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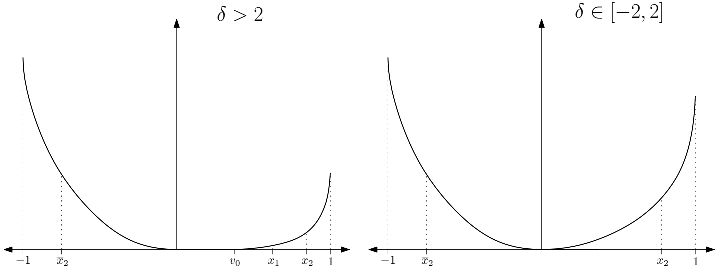

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   - $\delta \in [-2, 2]$: $I_X(x) = 0 \iff x = 0$.
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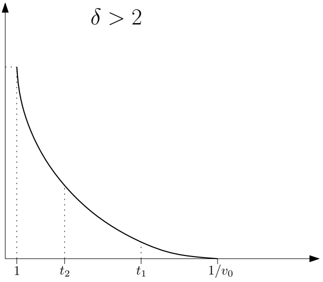
3. Derivatives
   - $I_X'(0) = \lim_{x \to 0} I_X(x)/x = 0$.

Properties of the rate function $I_T(t)$



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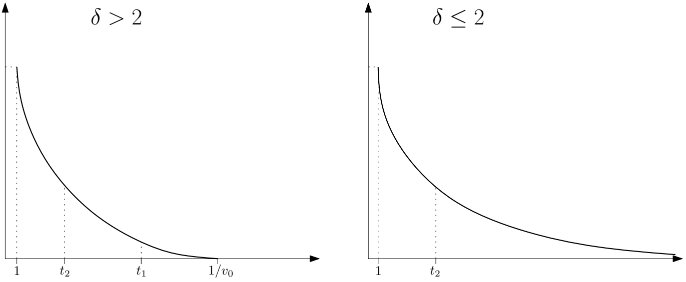
Properties of the rate function $I_T(t)$



1. $I_T(t)$ is a convex function.
2. Zero Set
   - $\delta \in [-2, 2]$: $I_T(t) > 0$ but $\lim_{t\to\infty} I_T(t) = 0$.
   - $\delta > 2$: $I_T(t) = 0 \iff t \geq 1/v_0$.

Slowdown probability asymptotics

$I_X(x) = 0 \iff x \in [0, v_0]$.
$P(X_n < xn)$ decays sub-exponentially for $x \in [0, v_0]$.

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Theorem (P. '12)

If $\delta > 2$, then

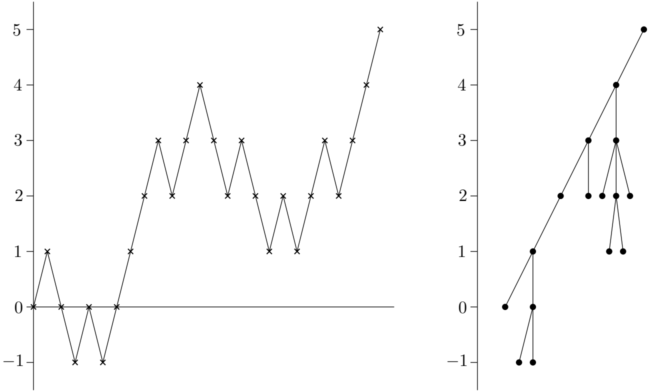
$$\lim_{n\to\infty} \frac{\log P(X_n < xn)}{\log n} = 1 - \frac{\delta}{2}, \qquad \forall x \in (0, v_0)$$

and

$$\lim_{n\to\infty} \frac{\log P(T_n > tn)}{\log n} = 1 - \frac{\delta}{2}, \qquad \forall t > 1/v_0.$$

Similar to slowdown asymptotics for RWRE.

Associated branching process with migration



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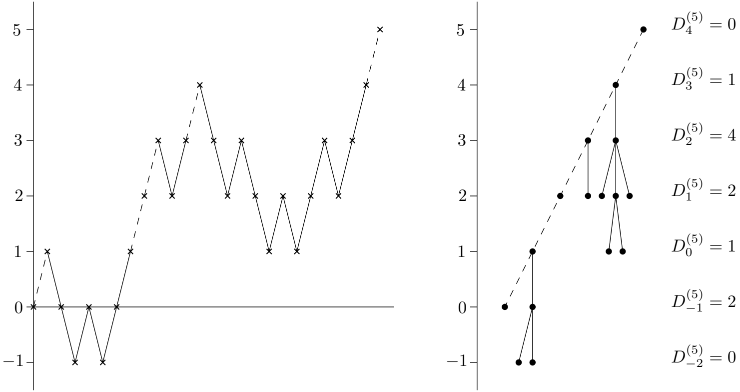

$$T_n = n + 2 \sum_{i<n} D_i^{(n)}$$

$$D_i^{(n)} = \#\text{ steps } \leftarrow \text{ from } i \text{ before } T_n$$

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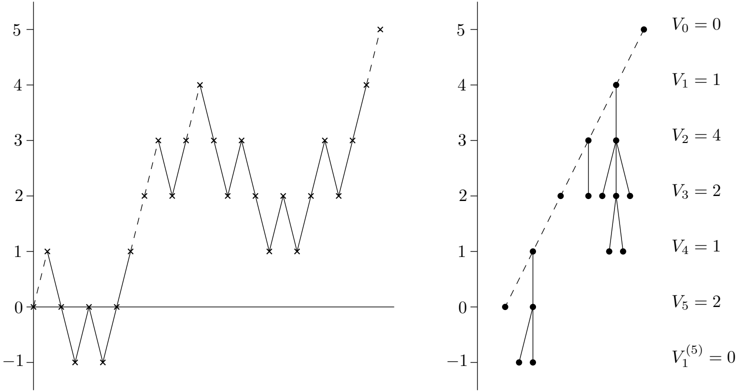

$$T_n = n + 2\sum_{i<n} D_i^{(n)} \overset{\text{Law}}{=} n + 2\sum_{i=1}^{n} V_i + 2\sum_{i=1}^{\infty} V_i^{(n)}$$

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