Heat Trace of Non-local Operators

Selma Yıldırım Yolcu

PURDUE UNIVERSITY

Joint work with Rodrigo Bañuelos

17th October 2012

Table of contents

Table of Contents

PURDUE UNIVERSITY

Aim

Aim

Compute several coefficients in the asymptotic expansion of the trace of the heat kernel of the Schrödinger operator $\Delta^{\alpha/2} + V$ as $t \downarrow 0$.

The main object of study is the trace difference

$$Tr(e^{-tH} - e^{-tH_0}),$$

where $H_0 = \Delta^{\alpha/2}$ and $H = \Delta^{\alpha/2} + V$.

Motivation

Asymptotic expansion of the trace of the heat kernel of the Schrödinger operator $-\Delta + V$ as $t \downarrow 0$.

- Lieb (1967) - 2nd virial coefficient of a hard-sphere gas at low temperatures
- Penrose- Penrose- Stell (1994) - on sticky spheres in quantum mechanics
- Datchev- Hezari (2011) - overview article, various other spectral asymptotic results and applications
- .... and many more...

PURDUE UNIVERSITY

Motivation

Asymptotic expansion of the trace of the heat kernel of the Schrödinger operator $-\Delta + V$ as $t \downarrow 0$.

- ⋆ Lieb (1967) - 2nd virial coefficient of a hard-sphere gas at low temperatures
- Penrose- Penrose- Stell (1994) - on sticky spheres in quantum mechanics
- Datchev- Hezari (2011) - overview article, various other spectral asymptotic results and applications
- .... and many more...

Motivation

Asymptotic expansion of the trace of the heat kernel of the Schrödinger operator $-\Delta + V$ as $t \downarrow 0$.

- Lieb (1967) - 2nd virial coefficient of a hard-sphere gas at low temperatures
- ⋆ Penrose- Penrose- Stell (1994) - on sticky spheres in quantum mechanics
- Datchev- Hezari (2011) - overview article, various other spectral asymptotic results and applications
- .... and many more...

Motivation

Asymptotic expansion of the trace of the heat kernel of the Schrödinger operator $-\Delta + V$ as $t \downarrow 0$.

- Lieb (1967) - 2nd virial coefficient of a hard-sphere gas at low temperatures
- Penrose- Penrose- Stell (1994) - on sticky spheres in quantum mechanics
- ⋆ Datchev- Hezari (2011) - overview article, various other spectral asymptotic results and applications
- .... and many more...

PURDUE UNIVERSITY

Motivation

Asymptotic expansion of the trace of the heat kernel of the Schrödinger operator $-\Delta + V$ as $t \downarrow 0$.

- Lieb (1967) - 2nd virial coefficient of a hard-sphere gas at low temperatures
- Penrose- Penrose- Stell (1994) - on sticky spheres in quantum mechanics
- Datchev- Hezari (2011) - overview article, various other spectral asymptotic results and applications

⋆ .... and many more...

PURDUE UNIVERSITY

Motivation

Asymptotic expansion of the trace of the heat kernel of the Schrödinger operator $-\Delta + V$ as $t \downarrow 0$.

- McKean-Moerbeke(1975) - For $d = 1$, a recurrent formula for the general coefficients in the expression using KdV methods.
- Colin de Verdière(1981) - first four coefficients for potentials in $\mathbb{R}^3$.
- Branson-Gilkey(1990) - heat coefficients for differential operators on manifolds.
- van den Berg (1991) - computation of the first two terms under Hölder continuity of the potential
- Bañuelos-Sa Barreto (1995) - Explicit formula for all the coefficients for $V \in \mathcal{S}(\mathbb{R}^d)$, the class of rapidly decaying functions at infinity.
- Donnely (2005) - extended to certain compact manifolds.
- Acuna Valverde (2012) - Asymptotic expansion of the trace of the heat kernel of the Schrödinger operator $(\Delta)^{\alpha/2} + V$ as $t \downarrow 0$.

Motivation

Asymptotic expansion of the trace of the heat kernel of the Schrödinger operator $-\Delta + V$ as $t \downarrow 0$.

- McKean-Moerbeke(1975) - For $d = 1$, a recurrent formula for the general coefficients in the expression using KdV methods.
- Colin de Verdière(1981) - first four coefficients for potentials in $\mathbb{R}^3$.
- Branson-Gilkey(1990) - heat coefficients for differential operators on manifolds.
- van den Berg (1991) - computation of the first two terms under Hölder continuity of the potential
- Bañuelos-Sa Barreto (1995) - Explicit formula for all the coefficients for $V \in \mathcal{S}(\mathbb{R}^d)$, the class of rapidly decaying functions at infinity.
- Donnely (2005) - extended to certain compact manifolds.
- Acuna Valverde (2012) - Asymptotic expansion of the trace of the heat kernel of the Schrödinger operator $(\Delta)^{\alpha/2} + V$ as $t \downarrow 0$.

Motivation

Asymptotic expansion of the trace of the heat kernel of the Schrödinger operator $-\Delta + V$ as $t \downarrow 0$.

- McKean-Moerbeke(1975) - For $d = 1$, a recurrent formula for the general coefficients in the expression using KdV methods.
- ⋆ Colin de Verdière(1981) - first four coefficients for potentials in $\mathbb{R}^3$.
- Branson-Gilkey(1990) - heat coefficients for differential operators on manifolds.
- van den Berg (1991) - computation of the first two terms under Hölder continuity of the potential
- Bañuelos-Sa Barreto (1995) - Explicit formula for all the coefficients for $V \in \mathcal{S}(\mathbb{R}^d)$, the class of rapidly decaying functions at infinity.
- Donnely (2005) - extended to certain compact manifolds.
- Acuna Valverde (2012) - Asymptotic expansion of the trace of the heat kernel of the Schrödinger operator $(\Delta)^{\alpha/2} + V$ as $t \downarrow 0$.

Motivation

Asymptotic expansion of the trace of the heat kernel of the Schrödinger operator $-\Delta + V$ as $t \downarrow 0$.

- McKean-Moerbeke(1975) - For $d = 1$, a recurrent formula for the general coefficients in the expression using KdV methods.
- Colin de Verdière(1981) - first four coefficients for potentials in $\mathbb{R}^3$.
- ⋆ Branson-Gilkey(1990) - heat coefficients for differential operators on manifolds.
- van den Berg (1991) - computation of the first two terms under Hölder continuity of the potential
- Bañuelos-Sa Barreto (1995) - Explicit formula for all the coefficients for $V \in \mathcal{S}(\mathbb{R}^d)$, the class of rapidly decaying functions at infinity.
- Donnely (2005) - extended to certain compact manifolds.
- Acuna Valverde (2012) - Asymptotic expansion of the trace of the heat kernel of the Schrödinger operator $(\Delta)^{\alpha/2} + V$ as $t \downarrow 0$.

Motivation

Asymptotic expansion of the trace of the heat kernel of the Schrödinger operator $-\Delta + V$ as $t \downarrow 0$.

- McKean-Moerbeke(1975) - For $d = 1$, a recurrent formula for the general coefficients in the expression using KdV methods.
- Colin de Verdière(1981) - first four coefficients for potentials in $\mathbb{R}^3$.
- Branson-Gilkey(1990) - heat coefficients for differential operators on manifolds.
- ⋆ van den Berg (1991) - computation of the first two terms under Hölder continuity of the potential
- Bañuelos-Sa Barreto (1995) - Explicit formula for all the coefficients for $V \in \mathcal{S}(\mathbb{R}^d)$, the class of rapidly decaying functions at infinity.
- Donnely (2005) - extended to certain compact manifolds.
- Acuna Valverde (2012) - Asymptotic expansion of the trace of the heat kernel of the Schrödinger operator $(\Delta)^{\alpha/2} + V$ as $t \downarrow 0$.

Motivation

Asymptotic expansion of the trace of the heat kernel of the Schrödinger operator $-\Delta + V$ as $t \downarrow 0$.

- McKean-Moerbeke(1975) - For $d = 1$, a recurrent formula for the general coefficients in the expression using KdV methods.
- Colin de Verdière(1981) - first four coefficients for potentials in $\mathbb{R}^3$.
- Branson-Gilkey(1990) - heat coefficients for differential operators on manifolds.
- van den Berg (1991) - computation of the first two terms under Hölder continuity of the potential
- ⋆ Bañuelos-Sa Barreto (1995) - Explicit formula for all the coefficients for $V \in \mathcal{S}(\mathbb{R}^d)$, the class of rapidly decaying functions at infinity.
- Donnely (2005) - extended to certain compact manifolds.
- Acuna Valverde (2012) - Asymptotic expansion of the trace of the heat kernel of the Schrödinger operator $(\Delta)^{\alpha/2} + V$ as $t \downarrow 0$.

Motivation

Asymptotic expansion of the trace of the heat kernel of the Schrödinger operator $-\Delta + V$ as $t \downarrow 0$.

- McKean-Moerbeke(1975) - For $d = 1$, a recurrent formula for the general coefficients in the expression using KdV methods.
- Colin de Verdière(1981) - first four coefficients for potentials in $\mathbb{R}^3$.
- Branson-Gilkey(1990) - heat coefficients for differential operators on manifolds.
- van den Berg (1991) - computation of the first two terms under Hölder continuity of the potential
- Bañuelos-Sa Barreto (1995) - Explicit formula for all the coefficients for $V \in \mathcal{S}(\mathbb{R}^d)$, the class of rapidly decaying functions at infinity.
- ⋆ Donnely (2005) - extended to certain compact manifolds.
- Acuna Valverde (2012) - Asymptotic expansion of the trace of the heat kernel of the Schrödinger operator $(\Delta)^{\alpha/2} + V$ as $t \downarrow 0$.

Motivation

Asymptotic expansion of the trace of the heat kernel of the Schrödinger operator $-\Delta + V$ as $t \downarrow 0$.

- McKean-Moerbeke(1975) - For $d = 1$, a recurrent formula for the general coefficients in the expression using KdV methods.
- Colin de Verdière(1981) - first four coefficients for potentials in $\mathbb{R}^3$.
- Branson-Gilkey(1990) - heat coefficients for differential operators on manifolds.
- van den Berg (1991) - computation of the first two terms under Hölder continuity of the potential
- Bañuelos-Sa Barreto (1995) - Explicit formula for all the coefficients for $V \in \mathcal{S}(\mathbb{R}^d)$, the class of rapidly decaying functions at infinity.
- Donnely (2005) - extended to certain compact manifolds.
- ⋆ Acuna Valverde (2012) - Asymptotic expansion of the trace of the heat kernel of the Schrödinger operator $(\Delta)^{\alpha/2} + V$ as $t \downarrow 0$.

Motivation

Let $H_0 = -\Delta$ and $H = H_0 + V$, $V \in \mathcal{S}(\mathbb{R}^d)$. Set $I_j = \{(\lambda_1, \dots, \lambda_j) : 1 > \lambda_1 > \lambda_2 > \dots > \lambda_j > 0\}$.

Theorem (Bañuelos-Sá Barreto(1995))

For any integer $N \geq 1$, as $t \downarrow 0$

$$\frac{Tr(e^{-tH} - e^{-tH_0})}{p_t^{(2)}(0)} = \sum_{m=1}^{N} c_m(V) t^m + \mathcal{O}(t^{N+1})$$

where

$$c_1(V) = -\int_{\mathbb{R}^d} V(\theta) d\theta,$$

$$c_m(V) = (-1)^m \sum_{j+n=m, j\geq 2} \frac{(2\pi)^d}{(2\pi)^{jd} n!} \int_{I_j} \int_{\mathbb{R}^{(j-1)d}} \left[A_j^n(\lambda, \theta) \hat{V}\left(-\sum_{i=1}^{j-1} \theta_i\right) \prod_{i=1}^{j-1} \hat{V}(\theta_i) d\theta_i d\lambda_i d\lambda_j \right]$$

with

$$A_j(\lambda, \theta) = \sum_{k=1}^{j-1} (\lambda_k - \lambda_{k+1}) \left| \sum_{i=1}^{k} \theta_i \right|^2 - \left| \sum_{k=1}^{j-1} (\lambda_k - \lambda_{k+1}) \sum_{i=1}^{k} \theta_i \right|^2.$$

Motivation

In particular, when $N = 2$, as $t \downarrow 0$, we have

$$\frac{Tr(e^{-tH} - e^{-tH_0})}{p_t^{(2)}(0)} + t\int_{\mathbb{R}^d} V(\theta)d\theta - \frac{t^2}{2!}\int_{\mathbb{R}^d} V^2(\theta)d\theta = \mathcal{O}(t^3),$$

which is the van den Berg(1993) result under the assumption on V.
When $N = 3$, as $t \downarrow 0$,

$$\begin{aligned}\frac{Tr(e^{-tH} - e^{-tH_0})}{p_t^{(2)}(0)} &+ t\int_{\mathbb{R}^d} V(\theta)d\theta - \frac{t^2}{2!}\int_{\mathbb{R}^d} V^2(\theta)d\theta \\ &+ \frac{t^3}{3!}\int_{\mathbb{R}^d} V^3(\theta)d\theta + \frac{t^3}{12}\int_{\mathbb{R}^d} |\nabla V(\theta)|^2 d\theta = \mathcal{O}(t^4).\end{aligned}$$

Motivation

In particular, when $N=2$, as $t \downarrow 0$, we have

$$\frac{Tr(e^{-tH}-e^{-tH_0})}{p_t^{(2)}(0)} + t\int_{\mathbb{R}^d} V(\theta)d\theta - \frac{t^2}{2!}\int_{\mathbb{R}^d} V^2(\theta)d\theta = \mathcal{O}(t^3),$$

which is the van den Berg(1993) result under the assumption on V.
When $N=3$, as $t \downarrow 0$,

$$\begin{aligned}\frac{Tr(e^{-tH}-e^{-tH_0})}{p_t^{(2)}(0)} &+ t\int_{\mathbb{R}^d} V(\theta)d\theta - \frac{t^2}{2!}\int_{\mathbb{R}^d} V^2(\theta)d\theta \\ &+ \frac{t^3}{3!}\int_{\mathbb{R}^d} V^3(\theta)d\theta + \frac{t^3}{12}\int_{\mathbb{R}^d} |\nabla V(\theta)|^2 d\theta = \mathcal{O}(t^4).\end{aligned}$$

Table of Contents

Lévy processes

A Lévy process is a stochastic process $X = (X_t),\ t \geq 0$ with

- ⋆ X has independent and stationary increments,
- $X_0 = 0$ (with probability 1),
- X is stochastically continuous: For all $\epsilon > 0$,

$$\lim_{t \to s} P\{|X_t - X_s| > \epsilon\} = 0.$$

Independent increments: The random variables $X_{t_1} - X_0$, $X_{t_2} - X_{t_1}$, ... , $X_{t_n} - X_{t_{n-1}}$ are independent for any given sequence of ordered times $0 < t_1 < t_2 < \cdots < t_n < \infty$.

Stationary increments: $0 < s < t < \infty$, $A \in \mathbb{R}^d$ Borel

$$P\{X_t - X_s \in A\} = P\{X_{t-s} \in A\}.$$

PURDUE UNIVERSITY

Lévy processes

A Lévy process is a stochastic process $X = (X_t),\ t \geq 0$ with

- X has independent and stationary increments,
- ⋆ $X_0 = 0$ (with probability 1),
- X is stochastically continuous: For all $\epsilon > 0$,

$$\lim_{t \to s} P\{|X_t - X_s| > \epsilon\} = 0.$$

Independent increments: The random variables $X_{t_1} - X_0$, $X_{t_2} - X_{t_1}$, ... , $X_{t_n} - X_{t_{n-1}}$ are independent for any given sequence of ordered times $0 < t_1 < t_2 < \cdots < t_n < \infty$.

Stationary increments: $0 < s < t < \infty$, $A \in \mathbb{R}^d$ Borel

$$P\{X_t - X_s \in A\} = P\{X_{t-s} \in A\}.$$

PURDUE UNIVERSITY

Lévy processes

A Lévy process is a stochastic process $X = (X_t),\ t \geq 0$ with

- X has independent and stationary increments,
- $X_0 = 0$ (with probability 1),
- ⋆ X is stochastically continuous: For all $\epsilon > 0$,

$$\lim_{t \to s} P\{|X_t - X_s| > \epsilon\} = 0.$$

Independent increments: The random variables $X_{t_1} - X_0$, $X_{t_2} - X_{t_1}$, ... , $X_{t_n} - X_{t_{n-1}}$ are independent for any given sequence of ordered times $0 < t_1 < t_2 < \cdots < t_n < \infty$.

Stationary increments: $0 < s < t < \infty$, $A \in \mathbb{R}^d$ Borel

$$P\{X_t - X_s \in A\} = P\{X_{t-s} \in A\}.$$

PURDUE UNIVERSITY

Lévy processes

A Lévy process is a stochastic process $X = (X_t),\ t \geq 0$ with

- X has independent and stationary increments,
- $X_0 = 0$ (with probability 1),
- X is stochastically continuous: For all $\epsilon > 0$,

$$\lim_{t \to s} P\{|X_t - X_s| > \epsilon\} = 0.$$

Independent increments: The random variables $X_{t_1} - X_0$, $X_{t_2} - X_{t_1}$, ... , $X_{t_n} - X_{t_{n-1}}$ are independent for any given sequence of ordered times $0 < t_1 < t_2 < \cdots < t_n < \infty$.

Stationary increments: $0 < s < t < \infty$, $A \in \mathbb{R}^d$ Borel

$$P\{X_t - X_s \in A\} = P\{X_{t-s} \in A\}.$$

PURDUE UNIVERSITY

Lévy processes

A Lévy process is a stochastic process $X = (X_t),\ t \geq 0$ with

- X has independent and stationary increments,
- $X_0 = 0$ (with probability 1),
- X is stochastically continuous: For all $\epsilon > 0$,

$$\lim_{t \to s} P\{|X_t - X_s| > \epsilon\} = 0.$$

Independent increments: The random variables $X_{t_1} - X_0$, $X_{t_2} - X_{t_1}$, ... , $X_{t_n} - X_{t_{n-1}}$ are independent for any given sequence of ordered times $0 < t_1 < t_2 < \cdots < t_n < \infty$.

Stationary increments: $0 < s < t < \infty$, $A \in \mathbb{R}^d$ Borel

$$P\{X_t - X_s \in A\} = P\{X_{t-s} \in A\}.$$

Lévy processes

The characteristic function of X_t is

$$\varphi_t(\xi) = E(e^{i\xi \cdot X_t}) = \int_{\mathbb{R}^d} e^{i\xi \cdot x} p_t(dx) = (2\pi)^{d/2} \widehat{p}_t(\xi)$$

where p_t is the distribution of X_t.

PURDUE UNIVERSITY

Stable processes

The rotationally invariant stable processes are self-similar processes, denoted by X_t^α with symbol $\rho(\xi) = -|\xi|^\alpha$, $0 < \alpha \leq 2$. That means,

$$\varphi_t(\xi) = E(e^{i\xi \cdot X_t^\alpha}) = e^{-t|\xi|^\alpha}.$$

Transition probabilities: For any Borel $A \subset \mathbb{R}^d$,

$$P^x\{X_t^\alpha \in A\} = \int_A p_t^{(\alpha)}(x-y)dy$$

where

$$p_t^{(\alpha)}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\xi\cdot x} e^{-t|\xi|^\alpha} d\xi.$$

PURDUE UNIVERSITY

Stable processes

The rotationally invariant stable processes are self-similar processes, denoted by X_t^α with symbol $\rho(\xi) = -|\xi|^\alpha$, $0 < \alpha \leq 2$. That means,

$$\varphi_t(\xi) = E(e^{i\xi \cdot X_t^\alpha}) = e^{-t|\xi|^\alpha}.$$

Transition probabilities: For any Borel $A \subset \mathbb{R}^d$,

$$P^x\{X_t^\alpha \in A\} = \int_A p_t^{(\alpha)}(x-y)dy$$

where

$$p_t^{(\alpha)}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} e^{-t|\xi|^\alpha} d\xi.$$

Stable processes

The rotationally invariant stable processes are self-similar processes, denoted by X_t^α with symbol $\rho(\xi) = -|\xi|^\alpha$, $0 < \alpha \leq 2$. That means,

$$\varphi_t(\xi) = E(e^{i\xi \cdot X_t^\alpha}) = e^{-t|\xi|^\alpha}.$$

Transition probabilities: For any Borel $A \subset \mathbb{R}^d$,

$$P^x\{X_t^\alpha \in A\} = \int_A p_t^{(\alpha)}(x-y)dy$$

where

$$p_t^{(\alpha)}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} e^{-t|\xi|^\alpha} d\xi.$$

Semigroup

For rapidly decaying functions $f \in \mathcal{S}(\mathbb{R}^d)$, we have the semigroup of the stable processes defined as

$$\begin{aligned} T_t f(x) &= E^x[f(X_t)] = E^0[f(X_t + x)] \\ &= \int_{\mathbb{R}^d} f(x+y) p_t(dy) = p_t * f(x) \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{-t|\xi|^\alpha} \widehat{f}(\xi) d\xi. \end{aligned}$$

PURDUE UNIVERSITY

Semigroup

For rapidly decaying functions $f \in \mathcal{S}(\mathbb{R}^d)$, we have the semigroup of the stable processes defined as

$$\begin{aligned} T_t f(x) &= E^x[f(X_t)] = E^0[f(X_t + x)] \\ &= \int_{\mathbb{R}^d} f(x+y) p_t(dy) = p_t * f(x) \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{-t|\xi|^\alpha} \widehat{f}(\xi) d\xi. \end{aligned}$$

By differentiating this at $t = 0$ we see that its infinitesimal generator is $\Delta^{\alpha/2}$ in the sense that $\widehat{\Delta^{\alpha/2} f}(\xi) = -|\xi|^\alpha \widehat{f}(\xi)$.

Semigroup

This is a non-local operator such that for suitable test functions, including all functions in $f \in C_0^\infty(\mathbb{R}^d)$, we can define it as the principle value integral

$$\Delta^{\alpha/2} f(x) = \mathcal{A}_{d,-\alpha} \lim_{\epsilon \to 0^+} \int_{\{|y|>\epsilon\}} \frac{f(x+y) - f(x)}{|y|^{d+\alpha}} dy,$$

where

$$\mathcal{A}_{d,-\alpha} = \frac{2^\alpha \Gamma\left(\frac{d+\alpha}{2}\right)}{\pi^{d/2} \left|\Gamma\left(\frac{-\alpha}{2}\right)\right|}.$$

Stable Processes-Examples

- Brownian motion $(\alpha = 2)$ has the transition density

$$p_t^{(2)}(x,y) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \exp\left(-\frac{|x-y|^2}{4t}\right), \qquad t > 0, \quad x, y \in \mathbb{R}^d.$$

- The infinitesimal generator of the Brownian motion for paths that are killed upon leaving the domain Ω is the Dirichlet Laplacian.

Stable Processes-Examples

- Brownian motion ($\alpha = 2$) has the transition density

$$p_t^{(2)}(x,y) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \exp\left(-\frac{|x-y|^2}{4t}\right), \qquad t > 0, \quad x, y \in \mathbb{R}^d.$$

- The infinitesimal generator of the Brownian motion for paths that are killed upon leaving the domain Ω is the Dirichlet Laplacian.

Stable Processes-Examples

- Cauchy process ($\alpha = 1$) has the transition density

$$p_t^{(1)}(x, y) = \frac{c_d\, t}{\left(t^2 + |x - y|^2\right)^{\frac{d+1}{2}}}, \qquad t > 0, \quad x, y \in \mathbb{R}^d$$

  where $c_d = \pi^{-\frac{d+1}{2}} \Gamma\left(\frac{d+1}{2}\right)$.
- The generator of the Cauchy process with the corresponding killing condition on $\partial\Omega$ is $\Delta^{1/2}|_{\Omega}$.

PURDUE UNIVERSITY

Stable processes -some properties

These processes share many of the basic properties of the Brownian motion:

- ⋆ $p_t^{(\alpha)}(x)$ is radial, symmetric and decreasing in x.
- Scaling: $p_t^{(\alpha)}(x,y) = t^{-d/\alpha} p_1^{(\alpha)}(t^{-1/\alpha}x, t^{-1/\alpha}y)$.
- $p_t^{(\alpha)}(x,y) = p_t^{(\alpha)}(x-y)$, in particular $p_t^{(\alpha)}(x,x) = p_t^{(\alpha)}(0)$.
- For all $x \in \mathbb{R}^d$ and $t > 0$,

$$C_{\alpha,d}^{-1}\left(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}\right) \le p_t^{(\alpha)}(x) \le C_{\alpha,d}\left(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}\right).$$

Here, $a \wedge b = \min\{a,b\}$ and $a \vee b = \max\{a,b\}$ for any $a, b \in \mathbb{R}^d$.

PURDUE UNIVERSITY

Stable processes -some properties

These processes share many of the basic properties of the Brownian motion:

- $p_t^{(\alpha)}(x)$ is radial, symmetric and decreasing in x.
- ⋆ Scaling: $p_t^{(\alpha)}(x, y) = t^{-d/\alpha} p_1^{(\alpha)}(t^{-1/\alpha}x, t^{-1/\alpha}y)$.
- $p_t^{(\alpha)}(x, y) = p_t^{(\alpha)}(x - y)$, in particular $p_t^{(\alpha)}(x, x) = p_t^{(\alpha)}(0)$.
- For all $x \in \mathbb{R}^d$ and $t > 0$,

$$C_{\alpha,d}^{-1}\left(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}\right) \leq p_t^{(\alpha)}(x) \leq C_{\alpha,d}\left(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}\right).$$

Here, $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$ for any $a, b \in \mathbb{R}^d$.

Stable processes -some properties

These processes share many of the basic properties of the Brownian motion:

- $p_t^{(\alpha)}(x)$ is radial, symmetric and decreasing in x.
- Scaling: $p_t^{(\alpha)}(x,y) = t^{-d/\alpha} p_1^{(\alpha)}(t^{-1/\alpha}x, t^{-1/\alpha}y)$.
- ⋆ $p_t^{(\alpha)}(x,y) = p_t^{(\alpha)}(x-y)$, in particular $p_t^{(\alpha)}(x,x) = p_t^{(\alpha)}(0)$.
- For all $x \in \mathbb{R}^d$ and $t > 0$,

$$C_{\alpha,d}^{-1}\left(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}\right) \leq p_t^{(\alpha)}(x) \leq C_{\alpha,d}\left(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}\right).$$

Here, $a \wedge b = \min\{a,b\}$ and $a \vee b = \max\{a,b\}$ for any $a, b \in \mathbb{R}^d$.

PURDUE UNIVERSITY

Stable processes -some properties

These processes share many of the basic properties of the Brownian motion:

- $p_t^{(\alpha)}(x)$ is radial, symmetric and decreasing in x.
- Scaling: $p_t^{(\alpha)}(x,y) = t^{-d/\alpha} p_1^{(\alpha)}(t^{-1/\alpha}x, t^{-1/\alpha}y)$.
- $p_t^{(\alpha)}(x,y) = p_t^{(\alpha)}(x-y)$, in particular $p_t^{(\alpha)}(x,x) = p_t^{(\alpha)}(0)$.

⋆ For all $x \in \mathbb{R}^d$ and $t > 0$,

$$C_{\alpha,d}^{-1}\left(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}\right) \le p_t^{(\alpha)}(x) \le C_{\alpha,d}\left(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}\right).$$

Here, $a \wedge b = \min\{a,b\}$ and $a \vee b = \max\{a,b\}$ for any $a, b \in \mathbb{R}^d$.

PURDUE UNIVERSITY

Table of Contents

Notation

- $H_0 = \Delta^{\alpha/2}, \alpha \in (0, 2]$ (the fractional Laplacian operator)
- e^{-tH_0}- the associated heat semigroup
- $p_t^{(\alpha)}$- transition density (heat kernel).

- $H = \Delta^{\alpha/2} + V$ (its Schrödinger perturbation), $V \in L^\infty(\mathbb{R}^d)$
- e^{-tH} - the associated heat semigroup
- p_t^H- transition density (heat kernel).

The Feynman-Kac formula gives

$$p_t^H(x, y) = p_t^{(\alpha)}(x, y) E_{x,y}^t \left(e^{-\int_0^t V(X_s) ds} \right),$$

where $E_{x,y}^t$ is the expectation with respect to the stable process (bridge) starting at x conditioned to be at y at time t.

Notation

- $H_0 = \Delta^{\alpha/2}, \alpha \in (0, 2]$ (the fractional Laplacian operator)
- e^{-tH_0}- the associated heat semigroup
- $p_t^{(\alpha)}$- transition density (heat kernel).

- $H = \Delta^{\alpha/2} + V$ (its Schrödinger perturbation), $V \in L^\infty(\mathbb{R}^d)$
- e^{-tH} - the associated heat semigroup
- p_t^H- transition density (heat kernel).

The Feynman-Kac formula gives

$$p_t^H(x, y) = p_t^{(\alpha)}(x, y) E_{x,y}^t \left(e^{-\int_0^t V(X_s) ds} \right),$$

where $E_{x,y}^t$ is the expectation with respect to the stable process (bridge) starting at x conditioned to be at y at time t.

Introduction

The main object of study is the trace difference

$$
\begin{aligned}
Tr(e^{-tH} - e^{-tH_0}) &= \int_{\mathbb{R}^d} (p_t^H(x,x) - p_t^{(\alpha)}(x,x))dx \\
&= p_t^{(\alpha)}(0) \int_{\mathbb{R}^d} E_{x,x}^t \left(e^{-\int_0^t V(X_s)ds} - 1 \right) dx \\
&= t^{-d/\alpha} p_1^{(\alpha)}(0) \int_{\mathbb{R}^d} E_{x,x}^t \left(e^{-\int_0^t V(X_s)ds} - 1 \right) dx,
\end{aligned}
$$

where $p_1^{(\alpha)}(0) = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha}$. Here, we denote by ω_d the surface area of the unit sphere in $\mathbb{R}^d$. This quantity is well defined for all $t > 0$, provided $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$.

Introduction

The main object of study is the trace difference

$$
\begin{aligned}
Tr(e^{-tH} - e^{-tH_0}) &= \int_{\mathbb{R}^d} (p_t^H(x,x) - p_t^{(\alpha)}(x,x))dx \\
&= p_t^{(\alpha)}(0) \int_{\mathbb{R}^d} E_{x,x}^t \left(e^{-\int_0^t V(X_s)ds} - 1 \right) dx \\
&= t^{-d/\alpha} p_1^{(\alpha)}(0) \int_{\mathbb{R}^d} E_{x,x}^t \left(e^{-\int_0^t V(X_s)ds} - 1 \right) dx,
\end{aligned}
$$

where $p_1^{(\alpha)}(0) = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha}$. Here, we denote by ω_d the surface area of the unit sphere in $\mathbb{R}^d$. This quantity is well defined for all $t > 0$, provided $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$.

Introduction

Indeed, the elementary inequality $|e^z - 1| \le |z|e^{|z|}$ immediately gives that

$$\left|\int_{\mathbb{R}^d} E^t_{x,x}\left(e^{-\int_0^t V(X_s)ds} - 1\right)dx\right| \le e^{t\|V\|_\infty}\int_{\mathbb{R}^d} E^t_{x,x}\left(\int_0^t |V(X_s)|ds\right)dx.$$

However,

$$\begin{aligned} E^t_{x,x}\left(\int_0^t |V(X_s)|ds\right) &= \int_0^t E^t_{x,x}|V(X_s)|ds \\ &= \int_0^t \int_{\mathbb{R}^d} \frac{p_s^{(\alpha)}(x,y)p_{t-s}^{(\alpha)}(y,x)}{p_t^{(\alpha)}(x,x)}|V(y)|dyds. \end{aligned}$$

Introduction

Chapman–Kolmogorov equations and the fact that $p_t^{(\alpha)}(x,x) = p_t^{(\alpha)}(0,0)$ give that

$$\int_{\mathbb{R}^d} \frac{p_s^{(\alpha)}(x,y)p_{t-s}^{(\alpha)}(y,x)}{p_t^{(\alpha)}(x,x)}\,dx = 1$$

and hence

$$\int_{\mathbb{R}^d} E_{x,x}^t \left(\int_0^t |V(X_s)| ds \right) dx = t\|V\|_1.$$

Introduction

It follows then that

$$\left| Tr(e^{-tH} - e^{-tH_0}) \right| \le t^{-d/\alpha+1} p_1^{(\alpha)}(0) \|V\|_1 e^{t\|V\|_\infty},$$

valid for all $t > 0$ and all potentials $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. The previous argument also shows that for all potentials $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$,

$$Tr\left(e^{-tH} - e^{-tH_0}\right) = p_t^{(\alpha)}(0) \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{\mathbb{R}^d} E_{x,x}^t \left(\int_0^t V(X_s)ds\right)^k dx,$$

where the sum is absolutely convergent for all $t > 0$.

Introduction

It follows then that

$$\left| Tr(e^{-tH} - e^{-tH_0}) \right| \le t^{-d/\alpha+1} p_1^{(\alpha)}(0) \|V\|_1 e^{t\|V\|_\infty},$$

valid for all $t > 0$ and all potentials $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. The previous argument also shows that for all potentials $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$,

$$Tr\left(e^{-tH} - e^{-tH_0}\right) = p_t^{(\alpha)}(0) \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{\mathbb{R}^d} E_{x,x}^t \left(\int_0^t V(X_s)ds\right)^k dx,$$

where the sum is absolutely convergent for all $t > 0$.

Theorem 1

Theorem (Bañuelos- Y.Y. (2012))

(i) Let $V : \mathbb{R}^d \to (-\infty, 0]$, $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. *Then for all* $t > 0$

$$\begin{aligned} p_t^\alpha(0) t \|V\|_1 &\leq Tr(e^{-tH} - e^{-tH_0}) \\ &\leq p_t^{(\alpha)}(0) \left(t\|V\|_1 + \frac{1}{2} t^2 \|V\|_1 \|V\|_\infty e^{t\|V\|_\infty} \right). \end{aligned}$$

In particular

$$\begin{aligned} Tr(e^{-tH} - e^{-tH_0}) &= p_t^{(\alpha)}(0) \left(t\|V\|_1 + \mathcal{O}(t^2) \right) \\ &= t^{-d/\alpha} p_1^{(\alpha)}(0) \left(t\|V\|_1 + \mathcal{O}(t^2) \right), \end{aligned}$$

as $t \downarrow 0$.

Theorem 1

Theorem (Bañuelos- Y.Y. (2012))

(ii) If we only assume that $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$*, then for all* $t > 0$,

$$\begin{aligned} \left| Tr(e^{-tH} - e^{-tH_0}) \right. &+ \left. p_t^{(\alpha)}(0) t \int_{\mathbb{R}^d} V(x)dx \right| \\ &\leq p_t^{(\alpha)}(0) C t^2 \|V\|_1 \|V\|_\infty e^{t\|V\|_\infty}, \end{aligned}$$

for some universal constant C*. From this we conclude that*

$$Tr(e^{-tH} - e^{-tH_0}) = p_t^{(\alpha)}(0) \left(-t \int_{\mathbb{R}^d} V(x)dx + \mathcal{O}(t^2) \right),$$

as $t \downarrow 0$.

Proof of Theorem 1

Setting

$$a = \int_0^t V(X_s)ds, \quad \text{and} \quad b = t\|V\|_\infty,$$

we observe that $-b \leq a \leq 0$. By using

$$-a \leq e^{-a} - 1 \leq -a\left(1 + \frac{1}{2}be^b\right)$$

we have

$$\begin{aligned} -\int_0^t V(X_s)ds &\leq \left(e^{-\int_0^t V(X_s)ds} - 1\right) \\ &\leq \left[-\int_0^t V(X_s)ds\right]\left(1 + \frac{1}{2}t\|V\|_\infty e^{t\|V\|_\infty}\right). \end{aligned}$$

Taking expectations of both sides of this inequality with respect to $E_{x,x}^t$ and then integrating on $\mathbb{R}^d$ with respect to x concludes the proof of (i) in Theorem 1.

PURDUE UNIVERSITY

Proof of Theorem 1

Setting

$$a = \int_0^t V(X_s)ds, \quad \text{and} \quad b = t\|V\|_\infty,$$

we observe that $-b \leq a \leq 0$. By using

$$-a \leq e^{-a} - 1 \leq -a\left(1 + \frac{1}{2}be^b\right)$$

we have

$$\begin{aligned} -\int_0^t V(X_s)ds &\leq \left(e^{-\int_0^t V(X_s)ds} - 1\right) \\ &\leq \left[-\int_0^t V(X_s)ds\right]\left(1 + \frac{1}{2}t\|V\|_\infty e^{t\|V\|_\infty}\right). \end{aligned}$$

Taking expectations of both sides of this inequality with respect to $E^t_{x,x}$ and then integrating on $\mathbb{R}^d$ with respect to x concludes the proof of (i) in Theorem 1.

PURDUE UNIVERSITY

Proof of Theorem 1

Setting

$$a = \int_0^t V(X_s)ds, \quad \text{and} \quad b = t\|V\|_\infty,$$

we observe that $-b \le a \le 0$. By using

$$-a \le e^{-a} - 1 \le -a\left(1 + \frac{1}{2}be^b\right)$$

we have

$$\begin{aligned} -\int_0^t V(X_s)ds &\le \left(e^{-\int_0^t V(X_s)ds} - 1\right) \\ &\le \left[-\int_0^t V(X_s)ds\right]\left(1 + \frac{1}{2}t\|V\|_\infty e^{t\|V\|_\infty}\right). \end{aligned}$$

Taking expectations of both sides of this inequality with respect to $E^t_{x,x}$ and then integrating on $\mathbb{R}^d$ with respect to x concludes the proof of (i) in Theorem 1.

Proof of Theorem 1

(ii) Observe that

$$\left| Tr(e^{-tH} - e^{-tH_0}) + p_t^{(\alpha)}(0)t \int_{\mathbb{R}^d} V(x)dx \right|$$

$$\leq p_t^{(\alpha)}(0) \sum_{k=2}^{\infty} \frac{1}{k!} \int_{\mathbb{R}^d} E_{x,x}^t \left| \int_0^t V(X_s)ds \right|^k dx$$

$$\leq p_t^{(\alpha)}(0) \sum_{k=2}^{\infty} \frac{t^{k-1}\|V\|_\infty^{k-1}}{k!} \int_{\mathbb{R}^d} E_{x,x} \left(\int_0^t |V(X_s)|ds \right) dx$$

$$= p_t^{(\alpha)}(0)t\|V\|_1 \sum_{k=2}^{\infty} \frac{t^{k-1}\|V\|_\infty^{k-1}}{k!} \leq Cp_t^{(\alpha)}(0)t^2\|V\|_1\|V\|_\infty e^{t\|V\|_\infty},$$

for some absolute constant C. This concludes the proof.

PURDUE UNIVERSITY

Theorem 2

Theorem (Bañuelos- Y.Y. (2012))

Suppose $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ and that it is also uniformly Hölder continuous of order γ (i.e., there exists a constant $M \in (0,\infty)$ such that $|V(x) - V(y)| \leq M|x-y|^\gamma$, for all $x, y \in \mathbb{R}^d$) with $0 < \gamma < \alpha \wedge 1$, whenever $0 < \alpha \leq 1$, and with $0 < \gamma \leq 1$, whenever $1 < \alpha < 2$. Then for all $t > 0$,

$$\left| \left(Tr(e^{-tH} - e^{-tH_0}) \right) + p_t^{(\alpha)}(0) t \int_{\mathbb{R}^d} V(x)dx - p_t^{(\alpha)}(0) \frac{1}{2} t^2 \int_{\mathbb{R}^d} |V(x)|^2 dx \right|$$
$$\leq C_{\alpha,\gamma,d} \|V\|_1 p_t^{(\alpha)}(0) \left(\|V\|_\infty^2 e^{t\|V\|_\infty} t^3 + t^{\gamma/\alpha+2} \right),$$

where the constant $C_{\alpha,\gamma,d}$ depends only on α, γ and d. In particular,

$$Tr(e^{-tH} - e^{-tH_0}) = p_t^{(\alpha)}(0) \left(-t \int_{\mathbb{R}^d} V(x)dx + \frac{1}{2} t^2 \int_{\mathbb{R}^d} |V(x)|^2 dx + \mathcal{O}(t^{\gamma/\alpha+2}) \right),$$

as $t \downarrow 0$.

Proof of Theorem 2

We begin by observing that we have

$$\left| e^{-\int_0^t V(X_s)ds} - 1 + \int_0^t V(X_s)ds - \frac{1}{2}\left[\int_0^t V(X_s)ds\right]^2 \right|$$

$$\leq \quad C(t\|V\|_\infty)^2 e^{t\|V\|_\infty} \int_0^t |V(X_s)|ds,$$

for some constant C. By taking expectation of both sides with respect to $E^t_{x,x}$ and then integrating with respect to x, we obtain

$$\int_{\mathbb{R}^d} E^t_{x,x}\left(\left| e^{-\int_0^t V(X_s)ds} - 1 + \int_0^t V(X_s)ds - \frac{1}{2}\left[\int_0^t V(X_s)ds\right]^2 \right|\right) dx$$

$$\leq \quad C(t\|V\|_\infty)^2 e^{t\|V\|_\infty} \int_{\mathbb{R}^d} E^t_{x,x}\left(\int_0^t |V(X_s)|ds\right) dx$$

$$= \quad C(t\|V\|_\infty)^2 e^{t\|V\|_\infty} t\|V\|_1.$$

PURDUE UNIVERSITY

Proof of Theorem 2

We begin by observing that we have

$$\begin{aligned} & \left| e^{-\int_0^t V(X_s)ds} - 1 + \int_0^t V(X_s)ds - \frac{1}{2}\left[\int_0^t V(X_s)ds\right]^2 \right| \\ \leq \quad & C(t\|V\|_\infty)^2 e^{t\|V\|_\infty} \int_0^t |V(X_s)|ds, \end{aligned}$$

for some constant C.By taking expectation of both sides with respect to $E^t_{x,x}$ and then integrating with respect to x, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^d} E^t_{x,x}\left(\left| e^{-\int_0^t V(X_s)ds} - 1 + \int_0^t V(X_s)ds - \frac{1}{2}\left[\int_0^t V(X_s)ds\right]^2 \right|\right) dx \\ \leq \quad & C(t\|V\|_\infty)^2 e^{t\|V\|_\infty} \int_{\mathbb{R}^d} E^t_{x,x}\left(\int_0^t |V(X_s)|ds\right) dx \\ = \quad & C(t\|V\|_\infty)^2 e^{t\|V\|_\infty} t\|V\|_1. \end{aligned}$$

Proof of Theorem 2

Returning to the definition of the trace differences, we see that this leads to

$$\left|\frac{1}{p_t^{(\alpha)}(0)}(Tr(e^{-tH}-e^{-tH_0}))+t\int_{\mathbb{R}^d}V(x)dx-\frac{1}{2}\int_{\mathbb{R}^d}E_{x,x}^t\left(\left[\int_0^t V(X_s)ds\right]^2\right)dx\right|$$
$$\leq C(t\|V\|_\infty)^2 e^{t\|V\|_\infty}t\|V\|_1.$$

It remains to estimate the term $E_{x,x}^t([\cdot]^2)$. Since V is uniformly Hölder with exponent γ and constant M, we have

$$|V(X_s+x)-V(x)|\leq M|X_s|^\gamma.$$

Proof of Theorem 2

Hence,

$$\begin{aligned}
\left| E_{x,x}^t \left[\int_0^t V(X_s) ds \right]^2 - t^2 V^2(x) \right| &= \left| E_{x,x}^t \left[\int_0^t V(X_s) ds \right]^2 - \left[\int_0^t V(x) ds \right]^2 \right| \\
&= \left| E_{x,x}^t \left(\left[\int_0^t V(X_s) ds \right]^2 - \left[\int_0^t V(x) ds \right]^2 \right) \right| \\
&= E_{0,0}^t \left(\left[\int_0^t (V(X_s + x) - V(x)) ds \right] \cdot \right. \\
&\quad \left. \left[\int_0^t V(X_s + x) + V(x)) ds \right] \right).
\end{aligned}$$

Proof of Theorem 2

Then

$$\left|E^t_{x,x}\left(\left[\int_0^t V(X_s)ds\right]^2\right) - t^2V^2(x)\right| \leq ME^t_{0,0}\left(\left[\int_0^t |X_s|^\gamma ds\right]\left[\int_0^t (|V(X_s+x)| + |V(x)|)\, ds\right]\right.$$

Integrating both sides of this inequality with respect to x and using Fubini's theorem, the second integral becomes $2t\|V\|_1$. Thus we arrive at

$$\left|\int_{\mathbb{R}^d} E^t_{x,x}\left(\left[\int_0^t V(X_s)ds\right]^2\right)dx - t^2\int_{\mathbb{R}^d}|V(x)|^2dx\right| \leq 2tM\|V\|_1 E^t_{0,0}\left(\int_0^t |X_s|^\gamma ds\right).$$

Proof of Theorem 2

Now, it remains to estimate the expectation on the right side. We have

$$
\begin{aligned}
E^t_{0,0}\left(\int_0^t |X_s|^\gamma ds\right) &= \int_0^t E^t_{0,0}(|X_s|^\gamma)ds \\
&= \int_0^t \int_{\mathbb{R}^d} \frac{p_s^{(\alpha)}(0,y)p_{t-s}^{(\alpha)}(y,0)}{p_t^{(\alpha)}(0,0)} |y|^\gamma dyds \\
&= \int_0^{t/2} \int_{\mathbb{R}^d} \frac{p_s^{(\alpha)}(0,y)p_{t-s}^{(\alpha)}(y,0)}{p_t^{(\alpha)}(0,0)} |y|^\gamma dyds \\
&+ \int_{t/2}^{t} \int_{\mathbb{R}^d} \frac{p_s^{(\alpha)}(0,y)p_{t-s}^{(\alpha)}(y,0)}{p_t^{(\alpha)}(0,0)} |y|^\gamma dyds \\
&= 2\int_0^{t/2} \int_{\mathbb{R}^d} \frac{p_s^{(\alpha)}(0,y)p_{t-s}^{(\alpha)}(y,0)}{p_t^{(\alpha)}(0,0)} |y|^\gamma dyds.
\end{aligned}
$$

Proof of Theorem 2

To estimate the right hand side we observe that for all $0 < s < t/2$ and all $y \in \mathbb{R}^d$,

$$p^{(\alpha)}_{t-s}(y,0) \le p^{(\alpha)}_{t-s}(0,0) \le p^{(\alpha)}_{t/2}(0,0).$$

By scaling

$$\frac{p^{(\alpha)}_{t/2}(0,0)}{p^{(\alpha)}_{t}(0,0)} = 2^{d/\alpha}$$

and therefore the right hand side is bounded above by

$$\begin{aligned} E^t_{0,0}\left(\int_0^t |X_s|^\gamma ds\right) &\le 2^{d/\alpha+1}\int_0^{t/2}\int_{\mathbb{R}^d} p^{(\alpha)}_s(0,y)|y|^\gamma dyds \\ &= 2^{d/\alpha+1}\int_0^{t/2} E^0(|X_s|^\gamma)ds \\ &= 2^{d/\alpha+1}\int_0^{t/2} s^{\gamma/\alpha}E^0(|X_1|^\gamma)ds \\ &= \frac{2^{d/\alpha+1}}{2^{\gamma/\alpha+1}}E^0(|X_1|^\gamma)\frac{t^{\gamma/\alpha+1}}{\gamma/\alpha+1}, \end{aligned}$$

Proof of Theorem 2

We now recall that $E^0(|X_1|^\gamma)$ is finite under our assumption that $\gamma < \alpha$. Thus we see that

$$E^t_{0,0}\left(\int_0^t |X_s|^\gamma ds\right) \leq C_{\alpha,\gamma,d}\, t^{\gamma/\alpha+1},$$

where the constant $C_{\alpha,\gamma,d}$ depends only on α, γ and d. We conclude that

$$\begin{aligned}\left|\int_{\mathbb{R}^d} E^t_{x,x}\left(\left[\int_0^t V(X_s)ds\right]^2\right) dx - t^2\int_{\mathbb{R}^d} |V(x)|^2 dx\right| &\leq 2tM\|V\|_1 E^t_{0,0}\left(\int_0^t |X_s|^\gamma ds\right)\\ &\leq M\|V\|_1 C_{\alpha,\gamma,d}\, t^{\gamma/\alpha+2}.\end{aligned}$$

Proof of Theorem 2

Then

$$\left|\frac{1}{p_t^{\alpha}(0)}\left(Tr(e^{-tH}-e^{-tH_0})\right)+t\int_{\mathbb{R}^d}V(x)dx-\frac{1}{2}t^2\int_{\mathbb{R}^d}|V(x)|^2dx\right|$$
$$\leq Ct^3\|V\|_\infty^2 e^{t\|V\|_\infty}\|V\|_1+M\|V\|_1 C_{\alpha,\gamma,d}\, t^{\gamma/\alpha+2}$$
$$\leq C_{\alpha,\gamma,d}\|V\|_1\left(\|V\|_\infty^2 e^{t\|V\|_\infty}t^3+t^{\gamma/\alpha+2}\right).$$

Rewriting this in the form stated in Theorem 2, we arrive at the announced bound

$$\left|\left(Tr(e^{-tH}-e^{-tH_0})\right)+p_t^{\alpha}(0)t\int_{\mathbb{R}^d}V(x)dx-p_t^{(\alpha)}(0)t^2\frac{1}{2}\int_{\mathbb{R}^d}|V(x)|^2dx\right|$$
$$\leq C_{\alpha,\gamma,d}\|V\|_1 p_t^{(\alpha)}(0)\left(\|V\|_\infty^2 e^{t\|V\|_\infty}t^3+t^{\gamma/\alpha+2}\right),$$

valid for all $t>0$.

Table of Contents

$H_0^a = \Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$, $a \geq 0$, $0 < \beta < \alpha < 2$

- Taking $0 < \beta < \alpha < 2$ and $a \geq 0$, consider the process $Z_t^a = X_t + aY_t$, where X_t and Y_t are independent α-stable and β-stable processes, respectively.
- This process is called the independent sum of the symmetric α-stable process X and the symmetric β-stable process Y with weight a.
- The infinitesimal generator of Z_t^a is $\Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$. Acting on functions $f \in C_0^\infty(\mathbb{R}^d)$ we have

$$\left(\Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}\right) f(x) =$$

$$\mathcal{A}_{d,-\alpha} \lim_{\epsilon \to 0^+} \int_{\{|y|>\epsilon\}} \left(\frac{1}{|y|^{d+\alpha}} + \frac{a^\beta}{|y|^{d+\beta}} \right) [f(x+y) - f(x)] dy,$$

where $\mathcal{A}_{d,-\alpha}$ is defined as before.

$H_0^a = \Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$, $a \geq 0$, $0 < \beta < \alpha < 2$

- Taking $0 < \beta < \alpha < 2$ and $a \geq 0$, consider the process $Z_t^a = X_t + aY_t$, where X_t and Y_t are independent α-stable and β-stable processes, respectively.

⋆ This process is called the independent sum of the symmetric α-stable process X and the symmetric β-stable process Y with weight a.

- The infinitesimal generator of Z_t^a is $\Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$. Acting on functions $f \in C_0^\infty(\mathbb{R}^d)$ we have

$$\left(\Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}\right) f(x) =$$

$$\mathcal{A}_{d,-\alpha} \lim_{\epsilon \to 0^+} \int_{\{|y|>\epsilon\}} \left(\frac{1}{|y|^{d+\alpha}} + \frac{a^\beta}{|y|^{d+\beta}}\right) [f(x+y) - f(x)] dy,$$

where $\mathcal{A}_{d,-\alpha}$ is defined as before.

$H_0^a = \Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$, $a \geq 0$, $0 < \beta < \alpha < 2$

- Taking $0 < \beta < \alpha < 2$ and $a \geq 0$, consider the process $Z_t^a = X_t + aY_t$, where X_t and Y_t are independent α-stable and β-stable processes, respectively.
- This process is called the independent sum of the symmetric α-stable process X and the symmetric β-stable process Y with weight a.

⋆ The infinitesimal generator of Z_t^a is $\Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$. Acting on functions $f \in C_0^\infty(\mathbb{R}^d)$ we have

$$\left(\Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}\right) f(x) =$$

$$\mathcal{A}_{d,-\alpha} \lim_{\epsilon \to 0^+} \int_{\{|y|>\epsilon\}} \left(\frac{1}{|y|^{d+\alpha}} + \frac{a^\beta}{|y|^{d+\beta}}\right) [f(x+y) - f(x)] dy,$$

where $\mathcal{A}_{d,-\alpha}$ is defined as before.

$$H_0^a = \Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$$

- For the properties of the heat kernel (transition probabilities) for this operator see Chen and Kumagai(2008), Chen-Kim-Song(2012) or Jakubowski-Szczypkowski(2011), and references given there.
- If we denote the heat kernel of this operator by $p_t^a(x)$, we have that

$$p_t^a(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-t(|\xi|^\alpha + a^\beta |\xi|^\beta)} d\xi = \int_0^\infty \frac{1}{(4\pi s)^{d/2}} e^{\frac{-|x|^2}{4s}} \eta_t^a(s)\, ds,$$

where $\eta_t^a(s)$ is be the density function of the sum of the $\alpha/2$-stable subordinator and a^2-times the $\beta/2$-stable subordinator.
- Again, this density is radial, symmetric, and decreasing in $|x|$.

PURDUE UNIVERSITY

$$H_0^a = \Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$$

- For the properties of the heat kernel (transition probabilities) for this operator see Chen and Kumagai(2008), Chen-Kim-Song(2012) or Jakubowski-Szczypkowski(2011), and references given there.

⋆ If we denote the heat kernel of this operator by $p_t^a(x)$, we have that

$$p_t^a(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-t(|\xi|^\alpha + a^\beta|\xi|^\beta)} d\xi = \int_0^\infty \frac{1}{(4\pi s)^{d/2}} e^{\frac{-|x|^2}{4s}} \eta_t^a(s)\, ds,$$

where $\eta_t^a(s)$ is be the density function of the sum of the $\alpha/2$-stable subordinator and a^2-times the $\beta/2$-stable subordinator.

- Again, this density is radial, symmetric, and decreasing in $|x|$.

PURDUE UNIVERSITY

$H_0^a = \Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$

- For the properties of the heat kernel (transition probabilities) for this operator see Chen and Kumagai(2008), Chen-Kim-Song(2012) or Jakubowski-Szczypkowski(2011), and references given there.
- If we denote the heat kernel of this operator by $p_t^a(x)$, we have that

$$p_t^a(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-t(|\xi|^\alpha + a^\beta |\xi|^\beta)} d\xi = \int_0^\infty \frac{1}{(4\pi s)^{d/2}} e^{\frac{-|x|^2}{4s}} \eta_t^a(s)\, ds,$$

where $\eta_t^a(s)$ is be the density function of the sum of the $\alpha/2$-stable subordinator and a^2-times the $\beta/2$-stable subordinator.

⋆ Again, this density is radial, symmetric, and decreasing in $|x|$.

PURDUE UNIVERSITY

$$H_0^a = \Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$$

⋆ There is a constant $C_{\alpha,\beta,d}$ such that for all $x \in \mathbb{R}^d$ and $t > 0$,

$$C_{\alpha,\beta,d}^{-1} f_t^a(x) \le p_t^a(x) \le C_{\alpha,\beta,d} f_t^a(x)$$

where

$$f_t^a(x) = \left((a^\beta t)^{-d/\beta} \wedge t^{-d/\alpha} \right) \wedge \left(\frac{t}{|x|^{d+\alpha}} + \frac{a^\beta t}{|x|^{d+\beta}} \right).$$

PURDUE UNIVERSITY

$$H_0^a = \Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$$

We note that for $a = 0$ this is just the estimate we obtained before. For any $\gamma > 0$ with $0 < \gamma < \beta < \alpha$ we have for any $t > 0$,

$$\begin{aligned}
\int_0^t E^0(|Z_s^a|^\gamma)ds &\leq C_\gamma \left(\int_0^t E^0(|X_s|^\gamma)ds + a^\gamma \int_0^t E^0(|Y_s|^\gamma)ds \right) \\
&= C_\gamma \left(E^0(|X_1|^\gamma) \frac{t^{\gamma/\alpha+1}}{\gamma/\alpha+1} + a^\gamma E^0(|Y_1|^\gamma) \frac{t^{\gamma/\beta+1}}{\gamma/\beta+1} \right) \\
&= C_{a,\alpha,\beta,d} \left(\frac{t^{\gamma/\alpha+1}}{\gamma/\alpha+1} + \frac{t^{\gamma/\beta+1}}{\gamma/\beta+1} \right).
\end{aligned}$$

$$H_0^a = \Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$$

Theorem (Bañuelos- Y.Y. (2012))

Let $a \geq 0$, $0 < \beta < \alpha < 2$ and let $H_0^a = \Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$. Suppose $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ and that it is also uniformly Hölder continuous of order γ, with $0 < \gamma < \beta \wedge 1$, whenever $0 < \beta \leq 1$, and with $0 < \gamma \leq 1$, whenever $1 < \beta < 2$. Let $H^a = \Delta^{\alpha/2} + a^\beta \Delta^{\beta/2} + V$. Then for all $t > 0$,

$$\left| \left(Tr(e^{-tH^a} - e^{-tH_0^a}) \right) + p_t^a(0) t \int_{\mathbb{R}^d} V(x)dx - p_t^a(0) \frac{1}{2} t^2 \int_{\mathbb{R}^d} |V(x)|^2 dx \right|$$
$$\leq C_{a,\alpha,\beta,\gamma,d} \|V\|_1 p_t^a(0) \left(\|V\|_\infty^2 e^{t\|V\|_\infty} t^3 + t^{\gamma/\alpha+2} + t^{\gamma/\beta+2} \right),$$

where the constant $C_{a,\alpha,\beta,\gamma,d}$ depends only on a, α, β, γ and d. In particular, as $t \downarrow 0$,

$$Tr(e^{-tH^a} - e^{-tH_0^a}) = p_t^a(0) \left(-t \int_{\mathbb{R}^d} V(x)dx + \frac{1}{2} t^2 \int_{\mathbb{R}^d} |V(x)|^2 dx + \mathcal{O}(t^{\gamma/\alpha+2}) \right).$$

α-stable relativistic process

This is again a Lévy process denote by X_t^m with characteristic function

$$e^{-t\left(\left(|\xi|^2+m^{2/\alpha}\right)^{\alpha/2}-m\right)} = E(e^{i\xi\cdot X_t^m}) = \int_{\mathbb{R}^d} e^{i\xi\cdot y} p_t^{(m,\alpha)}(y)dy,$$

for any $m \geq 0$ and $0 < \alpha < 2$. As in the case of stable processes, X_t^m is a subordination of Brownian motion and in fact

$$\begin{aligned} p_t^{(m,\alpha)}(x) = &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-t\left(\left(|\xi|^2+m^{2/\alpha}\right)^{\alpha/2}-m\right)} d\xi \\ &= \int_0^\infty \frac{1}{(4\pi s)^{d/2}} e^{\frac{-|x|^2}{4s}} \eta_t^{m,\alpha}(s)\, ds, \end{aligned}$$

where $\eta_t^{m,\alpha}(s)$ is the density of the subordinator with Bernstein function $\Phi(\lambda) = (\lambda + m^{2/\alpha})^{\alpha/2} - m$.

α-stable relativistic process

This is again a Lévy process denote by X_t^m with characteristic function

$$e^{-t\left(\left(|\xi|^2+m^{2/\alpha}\right)^{\alpha/2}-m\right)} = E(e^{i\xi\cdot X_t^m}) = \int_{\mathbb{R}^d} e^{i\xi\cdot y} p_t^{(m,\alpha)}(y)dy,$$

for any $m \geq 0$ and $0 < \alpha < 2$. As in the case of stable processes, X_t^m is a subordination of Brownian motion and in fact

$$\begin{aligned} p_t^{(m,\alpha)}(x) = &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-t\left(\left(|\xi|^2+m^{2/\alpha}\right)^{\alpha/2}-m\right)} d\xi \\ &= \int_0^\infty \frac{1}{(4\pi s)^{d/2}} e^{\frac{-|x|^2}{4s}} \eta_t^{m,\alpha}(s)\, ds, \end{aligned}$$

where $\eta_t^{m,\alpha}(s)$ is the density of the subordinator with Bernstein function $\Phi(\lambda) = (\lambda + m^{2/\alpha})^{\alpha/2} - m$.

Relativistic Brownian motion

- As before we see that $p_t^{(m,\alpha)}(x)$ is radial, symmetric, and decreasing in $|x|$.
- $p_t^{(m,\alpha)}(x) = m^{d/\alpha} p_{mt}^{(1,\alpha)}(m^{1/\alpha} x)$.
- Grzywny- Ryznar - (2008)

$$p_t^{(m,\alpha)}(x) = e^{mt} \int_0^\infty \frac{1}{(4\pi s)^{d/2}} e^{\frac{-|x|^2}{4s}} e^{-m^{2/\alpha} s} \eta_t^{\alpha/2}(s)\, ds,$$

where $\eta_t^{\alpha/2}(s)$ is the density for the $\alpha/2$-stable subordinator. By scaling

$$\eta_t^{\alpha/2}(s) = t^{-2/\alpha} \eta_1^{\alpha/2}(st^{-2/\alpha}).$$

- Hence changing variables leads to

$$\lim_{t\downarrow 0} e^{-mt} t^{d/\alpha} p_t^{(m,\alpha)}(0) = \int_0^\infty \frac{1}{(4\pi s)^{d/2}} \eta_1^{\alpha/2}(s)\, ds = p_1^{\alpha}(0) = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha},$$

PURDUE UNIVERSITY

Relativistic Brownian motion

- As before we see that $p_t^{(m,\alpha)}(x)$ is radial, symmetric, and decreasing in $|x|$.

$\star$ $p_t^{(m,\alpha)}(x) = m^{d/\alpha} p_{mt}^{(1,\alpha)}(m^{1/\alpha} x)$.

- Grzywny- Ryznar - (2008)

$$p_t^{(m,\alpha)}(x) = e^{mt} \int_0^\infty \frac{1}{(4\pi s)^{d/2}} e^{\frac{-|x|^2}{4s}} e^{-m^{2/\alpha}s} \eta_t^{\alpha/2}(s)\, ds,$$

where $\eta_t^{\alpha/2}(s)$ is the density for the $\alpha/2$-stable subordinator. By scaling

$$\eta_t^{\alpha/2}(s) = t^{-2/\alpha} \eta_1^{\alpha/2}(st^{-2/\alpha}).$$

- Hence changing variables leads to

$$\lim_{t \downarrow 0} e^{-mt} t^{d/\alpha} p_t^{(m,\alpha)}(0) = \int_0^\infty \frac{1}{(4\pi s)^{d/2}} \eta_1^{\alpha/2}(s)\, ds = p_1^\alpha(0) = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha},$$

PURDUE UNIVERSITY

Relativistic Brownian motion

- As before we see that $p_t^{(m,\alpha)}(x)$ is radial, symmetric, and decreasing in $|x|$.
- $p_t^{(m,\alpha)}(x) = m^{d/\alpha} p_{mt}^{(1,\alpha)}(m^{1/\alpha}x)$.

⋆ Grzywny- Ryznar - (2008)

$$p_t^{(m,\alpha)}(x) = e^{mt} \int_0^\infty \frac{1}{(4\pi s)^{d/2}} e^{\frac{-|x|^2}{4s}} e^{-m^{2/\alpha}s} \eta_t^{\alpha/2}(s)\, ds,$$

where $\eta_t^{\alpha/2}(s)$ is the density for the $\alpha/2$-stable subordinator. By scaling

$$\eta_t^{\alpha/2}(s) = t^{-2/\alpha} \eta_1^{\alpha/2}(st^{-2/\alpha}).$$

- Hence changing variables leads to

$$\lim_{t\downarrow 0} e^{-mt} t^{d/\alpha} p_t^{(m,\alpha)}(0) = \int_0^\infty \frac{1}{(4\pi s)^{d/2}} \eta_1^{\alpha/2}(s)\, ds = p_1^{\alpha}(0) = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha},$$

PURDUE UNIVERSITY

Relativistic Brownian motion

- As before we see that $p_t^{(m,\alpha)}(x)$ is radial, symmetric, and decreasing in $|x|$.
- $p_t^{(m,\alpha)}(x) = m^{d/\alpha} p_{mt}^{(1,\alpha)}(m^{1/\alpha} x)$.
- Grzywny- Ryznar - (2008)

$$p_t^{(m,\alpha)}(x) = e^{mt} \int_0^\infty \frac{1}{(4\pi s)^{d/2}} e^{\frac{-|x|^2}{4s}} e^{-m^{2/\alpha} s} \eta_t^{\alpha/2}(s)\, ds,$$

where $\eta_t^{\alpha/2}(s)$ is the density for the $\alpha/2$-stable subordinator. By scaling

$$\eta_t^{\alpha/2}(s) = t^{-2/\alpha} \eta_1^{\alpha/2}(st^{-2/\alpha}).$$

⋆ Hence changing variables leads to

$$\lim_{t \downarrow 0} e^{-mt} t^{d/\alpha} p_t^{(m,\alpha)}(0) = \int_0^\infty \frac{1}{(4\pi s)^{d/2}} \eta_1^{\alpha/2}(s)\, ds = p_1^{\alpha}(0) = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha},$$

PURDUE UNIVERSITY

Relativistic Brownian motion

- ⋆ The infinitesimal generator of X_t^m is given by $m - \left(m^{2/\alpha} - \Delta\right)^{\alpha/2}$.
- The case $\alpha = 1$ gives the generator $m - \sqrt{-\Delta + m^2}$ which is the free relativistic Hamiltonian. (see Carmona, Masters and Simon(1990))
- For estimates for the global transition probabilities $p_t^{(m,\alpha)}(x)$ and their Dirichlet counterparts for various domains, see Chen(2009), Chen-Song(2003) Chen-Kim-Kumagai(2011), Chen-Kim-Song(2012), Chen-Kim-Song(2012), Ryznar (2002), Grzywny-Ryznar (2008).

Relativistic Brownian motion

- The infinitesimal generator of X_t^m is given by $m - \left(m^{2/\alpha} - \Delta\right)^{\alpha/2}$.

⋆ The case $\alpha = 1$ gives the generator $m - \sqrt{-\Delta + m^2}$ which is the free relativistic Hamiltonian. (see Carmona, Masters and Simon(1990))

- For estimates for the global transition probabilities $p_t^{(m,\alpha)}(x)$ and their Dirichlet counterparts for various domains, see Chen(2009), Chen-Song(2003) Chen-Kim-Kumagai(2011), Chen-Kim-Song(2012), Chen-Kim-Song(2012), Ryznar (2002), Grzywny-Ryznar (2008).

Relativistic Brownian motion

- The infinitesimal generator of X_t^m is given by $m - \left(m^{2/\alpha} - \Delta\right)^{\alpha/2}$.
- The case $\alpha = 1$ gives the generator $m - \sqrt{-\Delta + m^2}$ which is the free relativistic Hamiltonian. (see Carmona, Masters and Simon(1990))

⋆ For estimates for the global transition probabilities $p_t^{(m,\alpha)}(x)$ and their Dirichlet counterparts for various domains, see Chen(2009), Chen-Song(2003) Chen-Kim-Kumagai(2011), Chen-Kim-Song(2012), Chen-Kim-Song(2012), Ryznar (2002), Grzywny-Ryznar (2008).

Relativistic Brownian motion

⋆ Chen, Kim and Song(2012) [Theorem 2.1]: For all $x \in \mathbb{R}^d$ and all $t \in (0,1]$,

$$C_{\alpha,m,d}^{-1}\, t^{-d/\alpha} \wedge \frac{t\,\Psi(m^{\frac{1}{\alpha}}|x|)}{|x|^{d+\alpha}} \leq p_t^{(m,\alpha)}(x) \leq C_{\alpha,m,d}\, t^{-d/\alpha} \wedge \frac{t\,\Psi(m^{\frac{1}{\alpha}}|x|)}{|x|^{d+\alpha}},$$

where

$$\Psi(r) = 2^{-(d+\alpha)}\Gamma\left(\frac{d+\alpha}{2}\right)^{-1} \int_0^\infty s^{\frac{d+\alpha}{2}-1} e^{-s/4} e^{-r^2/s} ds$$

which is a decreasing function of r^2 with $\Psi(0) = 1$, $\Psi(r) \leq 1$ and with

$$c_1^{-1} e^{-r} r^{(d+\alpha-1)/2} \leq \Psi(r) \leq c_1 e^{-r} r^{(d+\alpha-1)/2},$$

for all $r \geq 1$.

Relativistic Brownian motion

Theorem (Bañuelos- Y.Y. (2012))

Let $H_0^m = m - \left(m^{2/\alpha} - \Delta\right)^{\alpha/2}$. *Suppose* $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ *and that it is also uniformly Hölder continuous of order* γ, *with* $0 < \gamma < \alpha \wedge 1$, *whenever* $0 < \alpha \le 1$, *and with* $0 < \gamma \le 1$, *whenever* $1 < \alpha < 2$. *Let* $H^m = m - \left(m^{2/\alpha} - \Delta\right)^{\alpha/2} + V$. *Then for all* $t > 0$,

$$\left| \left(Tr(e^{-tH^m} - e^{-tH_0^m}) \right) + p_t^\alpha(0) t \int_{\mathbb{R}^d} V(x)dx - p_t^\alpha(0) \frac{1}{2} t^2 \int_{\mathbb{R}^d} |V(x)|^2 dx \right|$$
$$\le C_{\alpha,\gamma,m,d} \|V\|_1 p_t^{(m,\alpha)}(0) \left(\|V\|_\infty^2 e^{t\|V\|_\infty} t^3 + t^{\gamma/\alpha+2} \right),$$

In particular,

$$Tr(e^{-tH^m} - e^{-tH_0^m}) = p_t^{(m,\alpha)}(0) \left(-t \int_{\mathbb{R}^d} V(x)dx + \frac{1}{2} t^2 \int_{\mathbb{R}^d} |V(x)|^2 dx + \mathcal{O}(t^{\gamma/\alpha+2}) \right),$$

as $t \downarrow 0$.

Thank You!