Heat Trace of Non-local Operators

Selma Yıldırım Yolcu

Joint work with Rodrigo Bañuelos

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   - $H_0^a = \Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$, $a \geq 0$, $0 < \beta < \alpha < 2$
   - Relativistic Brownian motion
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Aim

Compute several coefficients in the asymptotic expansion of the trace of the heat kernel of the Schrödinger operator $\Delta^{\alpha/2} + V$ as $t \downarrow 0$.

The main object of study is the trace difference

$$Tr(e^{-tH} - e^{-tH_0}),$$

where $H_0 = \Delta^{\alpha/2}$ and $H = \Delta^{\alpha/2} + V$. 
Motivation

Asymptotic expansion of the trace of the heat kernel of the Schrödinger operator $-\Delta + V$ as $t \downarrow 0$.

- Lieb (1967) - 2nd virial coefficient of a hard-sphere gas at low temperatures
- Penrose- Penrose- Stell (1994) - on sticky spheres in quantum mechanics
- Datchev- Hezari (2011) - overview article, various other spectral asymptotic results and applications
- .... and many more...
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Heat Trace of Non-local Operators
Motivation

Let $H_0 = -\Delta$ and $H = H_0 + V$, $V \in \mathcal{S}(\mathbb{R}^d)$. Set $I_j = \{(\lambda_1, \ldots, \lambda_j) : 1 > \lambda_1 > \lambda_2 > \ldots > \lambda_j > 0\}$.

Theorem (Bañuelos-Sá Barreto(1995))

For any integer $N \geq 1$, as $t \downarrow 0$

$$
\frac{\text{Tr}(e^{-tH} - e^{-tH_0})}{p^{(2)}_t(0)} = \sum_{m=1}^{N} c_m(V) t^m + \mathcal{O}(t^{N+1})
$$

where

$$
c_1(V) = -\int_{\mathbb{R}^d} V(\theta) d\theta,
$$

$$
c_m(V) = (-1)^m \sum_{j+n=m, j \geq 2} \frac{(2\pi)^d}{(2\pi)^d n!} \int_{I_j} \int_{\mathbb{R}^{d-1}} \left[ A^n_j(\lambda, \theta) \hat{V} \left( -\sum_{i=1}^{j-1} \theta_i \right) \prod_{i=1}^{j-1} \hat{V}(\theta_i) d\theta_i d\lambda_i d\lambda_j \right]
$$

with

$$
A_j(\lambda, \theta) = \sum_{k=1}^{j-1} (\lambda_k - \lambda_{k+1}) \left| \sum_{i=1}^{k} \theta_i \right|^2 - \left| \sum_{k=1}^{j-1} (\lambda_k - \lambda_{k+1}) \sum_{i=1}^{k} \theta_i \right|^2 .
$$
Motivation

In particular, when $N = 2$, as $t \downarrow 0$, we have

$$\frac{\text{Tr}(e^{-tH} - e^{-tH_0})}{p_t^{(2)}(0)} + t \int_{\mathbb{R}^d} V(\theta) d\theta - \frac{t^2}{2!} \int_{\mathbb{R}^d} V^2(\theta) d\theta = O(t^3),$$

which is the van den Berg (1993) result under the assumption on $V$.

When $N = 3$, as $t \downarrow 0$,

$$\frac{\text{Tr}(e^{-tH} - e^{-tH_0})}{p_t^{(2)}(0)} + t \int_{\mathbb{R}^d} V(\theta) d\theta - \frac{t^2}{2!} \int_{\mathbb{R}^d} V^2(\theta) d\theta \quad + \quad t^3 \int_{\mathbb{R}^d} V^3(\theta) d\theta + \frac{t^3}{3!} \int_{\mathbb{R}^d} |\nabla V(\theta)|^2 d\theta = O(t^4).$$
Motivation

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$$+ \frac{t^3}{3!} \int_{\mathbb{R}^d} V^3(\theta)d\theta + \frac{t^3}{12} \int_{\mathbb{R}^d} |\nabla V(\theta)|^2d\theta = O(t^4).$$
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**Lévy processes**

A Lévy process is a stochastic process $X = (X_t)$, $t \geq 0$ with

- $X$ has **independent** and **stationary** increments,
- $X_0 = 0$ (with probability 1),
- $X$ is stochastically continuous: For all $\epsilon > 0$,

$$\lim_{t \to s} P\{|X_t - X_s| > \epsilon\} = 0.$$

**Independent increments:** The random variables $X_{t_1} - X_0$, $X_{t_2} - X_{t_1}$, ..., $X_{t_n} - X_{t_{n-1}}$ are independent for any given sequence of ordered times $0 < t_1 < t_2 < \cdots < t_n < \infty$.

**Stationary increments:** $0 < s < t < \infty$, $A \in \mathbb{R}^d$ Borel

$$P\{X_t - X_s \in A\} = P\{X_{t-s} \in A\}.$$
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Lévy processes

The characteristic function of $X_t$ is

$$
\varphi_t(\xi) = E(e^{i \xi \cdot X_t}) = \int_{\mathbb{R}^d} e^{i \xi \cdot x} p_t(dx) = (2\pi)^{d/2} \hat{p}_t(\xi)
$$

where $p_t$ is the distribution of $X_t$. 
Stable processes

The rotationally invariant stable processes are self-similar processes, denoted by $X_t^\alpha$ with symbol $\rho(\xi) = -|\xi|^\alpha$, $0 < \alpha \leq 2$. That means,

$$\varphi_t(\xi) = E(e^{i\xi \cdot X_t^\alpha}) = e^{-t|\xi|^\alpha}.$$ 

Transition probabilities: For any Borel $A \subset \mathbb{R}^d$,

$$P^x\{X_t^\alpha \in A\} = \int_A p_t^{(\alpha)}(x - y) dy$$

where

$$p_t^{(\alpha)}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} e^{-t|\xi|^\alpha} d\xi.$$
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Semigroup

For rapidly decaying functions $f \in \mathcal{S}(\mathbb{R}^d)$, we have the semigroup of the stable processes defined as

$$T_t f(x) = E^x[f(X_t)] = E^0[f(X_t + x)]$$

$$= \int_{\mathbb{R}^d} f(x + y) p_t(dy) = p_t * f(x)$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} e^{-t|\xi|^\alpha} \hat{f}(\xi) d\xi.$$
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$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle x,\xi \rangle} e^{-t|\xi|^\alpha} \hat{f}(\xi) d\xi.$$

By differentiating this at $t = 0$ we see that its infinitesimal generator is $\Delta^{\alpha/2}$ in the sense that $\hat{\Delta^{\alpha/2} f}(\xi) = -|\xi|^\alpha \hat{f}(\xi)$. 

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Heat Trace of Non-local Operators
This is a non-local operator such that for suitable test functions, including all functions in \( f \in C_0^\infty(\mathbb{R}^d) \), we can define it as the principle value integral

\[
\Delta^{\alpha/2} f(x) = A_{d,-\alpha} \lim_{\epsilon \to 0^+} \int_{\{|y| > \epsilon\}} \frac{f(x + y) - f(x)}{|y|^{d+\alpha}} dy,
\]

where

\[
A_{d,-\alpha} = \frac{2^\alpha \Gamma \left( \frac{d+\alpha}{2} \right)}{\pi^{d/2} \left| \Gamma \left( \frac{-\alpha}{2} \right) \right|}.
\]
Stable Processes-Examples

- Brownian motion \((\alpha = 2)\) has the transition density

\[
p_t^{(2)}(x, y) = \frac{1}{(4\pi t)^{d/2}} \exp \left( -\frac{|x - y|^2}{4t} \right), \quad t > 0, \quad x, y \in \mathbb{R}^d.
\]

- The infinitesimal generator of the Brownian motion for paths that are killed upon leaving the domain \(\Omega\) is the Dirichlet Laplacian.
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- Cauchy process \((\alpha = 1)\) has the transition density

\[
p^{(1)}_t(x, y) = \frac{c_d t}{(t^2 + |x - y|^2)^{\frac{d+1}{2}}}, \quad t > 0, \quad x, y \in \mathbb{R}^d
\]

where \(c_d = \pi^{-\frac{d+1}{2}} \Gamma \left(\frac{d+1}{2}\right)\).

- The generator of the Cauchy process with the corresponding killing condition on \(\partial \Omega\) is \(\Delta^{1/2}|_{\Omega}\).
Stable processes - some properties

These processes share many of the basic properties of the Brownian motion:

- \( p_t^{(\alpha)}(x) \) is radial, symmetric and decreasing in \( x \).
- Scaling: \( p_t^{(\alpha)}(x, y) = t^{-d/\alpha} p_1^{(\alpha)}(t^{-1/\alpha} x, t^{-1/\alpha} y) \).
- \( p_t^{(\alpha)}(x, y) = p_t^{(\alpha)}(x - y) \), in particular \( p_t^{(\alpha)}(x, x) = p_t^{(\alpha)}(0) \).
- For all \( x \in \mathbb{R}^d \) and \( t > 0 \),
  \[
  C_{\alpha,d}^{-1} \left( t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}} \right) \leq p_t^{(\alpha)}(x) \leq C_{\alpha,d} \left( t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}} \right).
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Here, \( a \wedge b = \min\{a, b\} \) and \( a \vee b = \max\{a, b\} \) for any \( a, b \in \mathbb{R}^d \).
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   - Relativistic Brownian motion
Notation

- $H_0 = \Delta^{\alpha/2}$, $\alpha \in (0, 2]$ (the fractional Laplacian operator)
- $e^{-tH_0}$ - the associated heat semigroup
- $p_{t}^{(\alpha)}$ - transition density (heat kernel).

- $H = \Delta^{\alpha/2} + V$ (its Schrödinger perturbation), $V \in L^\infty(\mathbb{R}^d)$
- $e^{-tH}$ - the associated heat semigroup
- $p_{t}^{H}$ - transition density (heat kernel).

The Feynman-Kac formula gives

$$p_{t}^{H}(x, y) = p_{t}^{(\alpha)}(x, y)E_{x,y}^{t} \left( e^{-\int_{0}^{t} V(X_{s})ds} \right),$$

where $E_{x,y}^{t}$ is the expectation with respect to the stable process (bridge) starting at $x$ conditioned to be at $y$ at time $t$. 
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Introduction

The main object of study is the trace difference

\[
Tr(e^{-tH} - e^{-tH_0}) = \int_{\mathbb{R}^d} (p_t^H(x, x) - p_t^{(\alpha)}(x, x)) dx
\]

\[
= p_t^{(\alpha)}(0) \int_{\mathbb{R}^d} E_{x,x}^t \left( e^{-\int_0^t V(X_s) ds} - 1 \right) dx
\]

\[
= t^{-d/\alpha} p_1^{(\alpha)}(0) \int_{\mathbb{R}^d} E_{x,x}^t \left( e^{-\int_0^t V(X_s) ds} - 1 \right) dx,
\]

where \( p_1^{(\alpha)}(0) = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^{d/\alpha}} \). Here, we denote by \( \omega_d \) the surface area of the unit sphere in \( \mathbb{R}^d \). This quantity is well defined for all \( t > 0 \), provided \( V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \).
The main object of study is the trace difference

\[ \text{Tr}(e^{-tH} - e^{-tH_0}) = \int_{\mathbb{R}^d} (p_t^H(x, x) - p_t^{(\alpha)}(x, x)) \, dx \]

\[ = p_t^{(\alpha)}(0) \int_{\mathbb{R}^d} E_{x,x}^t \left( e^{-\int_0^t V(X_s) \, ds} - 1 \right) \, dx \]

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Introduction

Indeed, the elementary inequality $|e^z - 1| \leq |z|e^{|z|}$ immediately gives that

$$\left| \int_{\mathbb{R}^d} E^t_{x,x} \left( e^{-\int_0^t V(X_s) ds} - 1 \right) dx \right| \leq e^{t \|V\|_{\infty}} \int_{\mathbb{R}^d} E^t_{x,x} \left( \int_0^t |V(X_s)| ds \right) dx.$$ 

However,

$$E^t_{x,x} \left( \int_0^t |V(X_s)| ds \right) = \int_0^t E^t_{x,x} |V(X_s)| ds$$

$$= \int_0^t \int_{\mathbb{R}^d} \frac{p^\alpha_s(x, y)p^\alpha_{t-s}(y, x)}{p^\alpha_t(x, x)} |V(y)| dy ds.$$
Chapman–Kolmogorov equations and the fact that $p_t^{(\alpha)}(x, x) = p_t^{(\alpha)}(0, 0)$ give that

$$\int_{\mathbb{R}^d} \frac{p_s^{(\alpha)}(x, y)p_{t-s}^{(\alpha)}(y, x)}{p_t^{(\alpha)}(x, x)} \, dx = 1$$

and hence

$$\int_{\mathbb{R}^d} E_{x,x}^t \left( \int_0^t |V(X_s)| \, ds \right) \, dx = t \| V \|_1.$$
Introduction

It follows then that

$$|Tr(e^{-tH} - e^{-tH_0})| \leq t^{-d/\alpha+1} p_1^{(\alpha)}(0) \|V\|_1 e^t \|V\|_{\infty},$$

valid for all $t > 0$ and all potentials $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. The previous argument also shows that for all potentials $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$,

$$Tr \left( e^{-tH} - e^{-tH_0} \right) = p_t^{(\alpha)}(0) \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{\mathbb{R}^d} E_{x,x}^t \left( \int_0^t V(X_s) ds \right)^k dx,$$

where the sum is absolutely convergent for all $t > 0$. 
Introduction

It follows then that

\[ |Tr(e^{-tH} - e^{-tH_0})| \leq t^{-d/\alpha+1} p_1^{(\alpha)}(0) \| V \|_1 e^t \| V \|_\infty, \]

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where the sum is absolutely convergent for all \( t > 0 \).
Theorem 1

Theorem (Bañuelos- Y.Y. (2012))

(i) Let $V : \mathbb{R}^d \rightarrow (-\infty, 0]$, $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Then for all $t > 0$

$$p_t^{\alpha}(0)t\|V\|_1 \leq \text{Tr}(e^{-tH} - e^{-tH_0}) \leq p_t^{(\alpha)}(0) \left( t\|V\|_1 + \frac{1}{2}t^2\|V\|_1\|V\|_\infty e^{t\|V\|_\infty} \right).$$

In particular

$$\text{Tr}(e^{-tH} - e^{-tH_0}) = p_t^{(\alpha)}(0) \left( t\|V\|_1 + O(t^2) \right) = t^{-d/\alpha}p_1^{(\alpha)}(0) \left( t\|V\|_1 + O(t^2) \right),$$

as $t \downarrow 0$. 

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Theorem 1

Theorem (Bañuelos- Y.Y. (2012))

(ii) If we only assume that \( V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \), then for all \( t > 0 \),

\[
\left| \text{Tr}(e^{-tH} - e^{-tH_0}) + p_t^{(\alpha)}(0) t \int_{\mathbb{R}^d} V(x) dx \right| \\
\leq p_t^{(\alpha)}(0) C t^2 \| V \|_1 \| V \|_{\infty} e^{t \| V \|_{\infty}},
\]

for some universal constant \( C \). From this we conclude that

\[
\text{Tr}(e^{-tH} - e^{-tH_0}) = p_t^{(\alpha)}(0) \left( -t \int_{\mathbb{R}^d} V(x) dx + O(t^2) \right),
\]

as \( t \downarrow 0 \).
Proof of Theorem 1

Setting

$$a = \int_0^t V(X_s)ds, \quad \text{and} \quad b = t \| V \|_\infty,$$

we observe that $$-b \leq a \leq 0.$$ By using

$$-a \leq e^{-a} - 1 \leq -a \left(1 + \frac{1}{2} be^b\right)$$

we have

$$- \int_0^t V(X_s)ds \leq \left(e^{- \int_0^t V(X_s)ds} - 1\right)$$

$$\leq \left(- \int_0^t V(X_s)ds\right) \left(1 + \frac{1}{2} t \| V \|_\infty e^{t \| V \|_\infty}\right).$$

Taking expectations of both sides of this inequality with respect to $$E_{x,x}^t$$ and then integrating on $$\mathbb{R}^d$$ with respect to $$x$$ concludes the proof of (i) in Theorem 1.
Proof of Theorem 1

Setting

\[ a = \int_0^t V(X_s)ds, \quad \text{and} \quad b = t\|V\|_\infty, \]

we observe that \(-b \leq a \leq 0\). By using

\[ -a \leq e^{-a} - 1 \leq -a \left( 1 + \frac{1}{2}be^b \right) \]

we have

\[ -\int_0^t V(X_s)ds \leq \left( e^{-\int_0^t V(X_s)ds} - 1 \right) \leq \left[ -\int_0^t V(X_s)ds \right] \left( 1 + \frac{1}{2}t\|V\|_\infty e^{t\|V\|_\infty} \right). \]

Taking expectations of both sides of this inequality with respect to \(E_{x,x}^t\) and then integrating on \(\mathbb{R}^d\) with respect to \(x\) concludes the proof of (i) in Theorem 1.
Proof of Theorem 1

Setting

\[ a = \int_0^t V(X_s) \, ds, \quad \text{and} \quad b = t \| V \|_{\infty}, \]

we observe that \(-b \leq a \leq 0\). By using

\[ -a \leq e^{-a} - 1 \leq -a \left( 1 + \frac{1}{2} be^b \right) \]

we have

\[ -\int_0^t V(X_s) \, ds \leq \left( e^{-\int_0^t V(X_s) \, ds} - 1 \right) \]

\[ \leq \left[ -\int_0^t V(X_s) \, ds \right] \left( 1 + \frac{1}{2} t \| V \|_{\infty} e^{t \| V \|_{\infty}} \right). \]

Taking expectations of both sides of this inequality with respect to \(E_{x,x}^t\) and then integrating on \(\mathbb{R}^d\) with respect to \(x\) concludes the proof of (i) in Theorem 1.
Proof of Theorem 1

(ii) Observe that

\[
\left| \operatorname{Tr}(e^{-tH} - e^{-tH_0}) + p_t^{(\alpha)}(0) t \int_{\mathbb{R}^d} V(x) dx \right|
\]

\[
\leq p_t^{(\alpha)}(0) \sum_{k=2}^{\infty} \frac{1}{k!} \int_{\mathbb{R}^d} E_{x,x}^t \left| \int_0^t V(X_s) ds \right|^k dx
\]

\[
\leq p_t^{(\alpha)}(0) \sum_{k=2}^{\infty} \frac{t^{k-1} \| V \|_{\infty}^{k-1}}{k!} \int_{\mathbb{R}^d} E_{x,x} \left( \int_0^t |V(X_s)| ds \right) dx
\]

\[
= p_t^{(\alpha)}(0) t \| V \|_1 \sum_{k=2}^{\infty} \frac{t^{k-1} \| V \|_{\infty}^{k-1}}{k!} \leq C p_t^{(\alpha)}(0) t^2 \| V \|_1 \| V \|_{\infty} e^{t \| V \|_{\infty}},
\]

for some absolute constant \( C \). This concludes the proof.
Theorem 2

Suppose $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ and that it is also uniformly Hölder continuous of order $\gamma$ (i.e., there exists a constant $M \in (0, \infty)$ such that $|V(x) - V(y)| \leq M|x - y|^{\gamma}$, for all $x, y \in \mathbb{R}^d$) with $0 < \gamma < \alpha \wedge 1$, whenever $0 < \alpha \leq 1$, and with $0 < \gamma \leq 1$, whenever $1 < \alpha < 2$. Then for all $t > 0$,

$$
\left| \left( \text{Tr}(e^{-tH} - e^{-tH_0}) \right) + p_t^{(\alpha)}(0) t \int_{\mathbb{R}^d} V(x) dx - p_t^{(\alpha)}(0) \frac{1}{2} t^2 \int_{\mathbb{R}^d} |V(x)|^2 dx \right|
\leq C_{\alpha, \gamma, d} \|V\|_1 p_t^{(\alpha)}(0) \left( \|V\|_\infty^2 e^t \|V\|_\infty t^3 + t^{\gamma/\alpha+2} \right),
$$

where the constant $C_{\alpha, \gamma, d}$ depends only on $\alpha$, $\gamma$ and $d$. In particular,

$$
\text{Tr}(e^{-tH} - e^{-tH_0}) = p_t^{(\alpha)}(0) \left( -t \int_{\mathbb{R}^d} V(x) dx + \frac{1}{2} t^2 \int_{\mathbb{R}^d} |V(x)|^2 dx + O(t^{\gamma/\alpha+2}) \right),
$$
as $t \downarrow 0$. 

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Heat Trace of Non-local Operators
Proof of Theorem 2

We begin by observing that we have

$$\left| e^{-\int_0^t V(X_s)ds} - 1 + \int_0^t V(X_s)ds - \frac{1}{2} \left[ \int_0^t V(X_s)ds \right]^2 \right| \leq C(t\|V\|_\infty)^2 e^{t\|V\|_\infty} \int_0^t |V(X_s)|ds,$$

for some constant $C$. By taking expectation of both sides with respect to $E_{x,x}^t$ and then integrating with respect to $x$, we obtain

$$\int_{\mathbb{R}^d} E_{x,x}^t \left( \left| e^{-\int_0^t V(X_s)ds} - 1 + \int_0^t V(X_s)ds - \frac{1}{2} \left[ \int_0^t V(X_s)ds \right]^2 \right| \right) dx$$

$$\leq C(t\|V\|_\infty)^2 e^{t\|V\|_\infty} \int_{\mathbb{R}^d} E_{x,x}^t \left( \int_0^t |V(X_s)|ds \right) dx$$

$$= C(t\|V\|_\infty)^2 e^{t\|V\|_\infty} t\|V\|_1.$$
Proof of Theorem 2

We begin by observing that we have

\[
\left| e^{- \int_0^t V(X_s)ds} - 1 + \int_0^t V(X_s)ds - \frac{1}{2} \left[ \int_0^t V(X_s)ds \right]^2 \right| \\
\leq C(t \| V \|_{\infty})^2 e^{t \| V \|_{\infty}} \int_0^t |V(X_s)| ds,
\]

for some constant \( C \).

By taking expectation of both sides with respect to \( E^t_{x,x} \) and then integrating with respect to \( x \), we obtain

\[
\int_{\mathbb{R}^d} E^t_{x,x} \left( \left| e^{- \int_0^t V(X_s)ds} - 1 + \int_0^t V(X_s)ds - \frac{1}{2} \left[ \int_0^t V(X_s)ds \right]^2 \right| \right) dx \\
\leq C(t \| V \|_{\infty})^2 e^{t \| V \|_{\infty}} \int_{\mathbb{R}^d} E^t_{x,x} \left( \int_0^t |V(X_s)| ds \right) dx \\
= C(t \| V \|_{\infty})^2 e^{t \| V \|_{\infty}} t \| V \|_1.
\]
Proof of Theorem 2

Returning to the definition of the trace differences, we see that this leads to

\[
\left| \frac{1}{p_t^{(\alpha)}(0)} (Tr(e^{-tH} - e^{-tH_0})) + t \int_{\mathbb{R}^d} V(x) dx - \frac{1}{2} \int_{\mathbb{R}^d} E_{x,x}^t \left( \left[ \int_0^t V(X_s) ds \right]^2 \right) dx \right|
\leq C(t\|V\|_{\infty})^2 e^{t\|V\|_{\infty}} t\|V\|_1.
\]

It remains to estimate the term \( E_{x,x}^t ([\cdot]^2) \). Since \( V \) is uniformly Hölder with exponent \( \gamma \) and constant \( M \), we have

\[
|V(X_s + x) - V(x)| \leq M|X_s|^{\gamma}.
\]
Proof of Theorem 2

Hence,

\[
\left| E_{x,x}^t \left[ \int_0^t V(X_s)ds \right]^2 - t^2 V^2(x) \right| = \left| E_{x,x}^t \left[ \int_0^t V(X_s)ds \right]^2 - \left[ \int_0^t V(x)ds \right]^2 \right|
\]

\[
= \left| E_{x,x}^t \left( \left[ \int_0^t V(X_s)ds \right]^2 - \left[ \int_0^t V(x)ds \right]^2 \right) \right|
\]

\[
= E_{0,0}^t \left( \left[ \int_0^t (V(X_s + x) - V(x))ds \right] \cdot \left[ \int_0^t V(X_s + x) + V(x)ds \right] \right).
\]
Proof of Theorem 2

Then

$$\left| E_{x,x}^t \left( \left[ \int_0^t V(X_s)ds \right]^2 \right) - t^2 V^2(x) \right| \leq ME_{0,0}^t \left( \left[ \int_0^t |X_s|^\gamma ds \right] \left[ \int_0^t (|V(X_s + x)| + |V(x)|) ds \right] \right)$$

Integrating both sides of this inequality with respect to \(x\) and using Fubini’s theorem, the second integral becomes \(2t \| V \|_1\). Thus we arrive at

$$\left| \int_{\mathbb{R}^d} E_{x,x}^t \left( \left[ \int_0^t V(X_s)ds \right]^2 \right) dx - t^2 \int_{\mathbb{R}^d} |V(x)|^2 dx \right| \leq 2tM \| V \|_1 E_{0,0}^t \left( \int_0^t |X_s|^\gamma ds \right).$$
Proof of Theorem 2

Now, it remains to estimate the expectation on the right side. We have

\[
E_{0,0}^{t} \left( \int_{0}^{t} |X_s|^{\gamma} ds \right) = \int_{0}^{t} E_{0,0}^{t}(|X_s|^{\gamma}) ds
\]

\[
= \int_{0}^{t} \int_{\mathbb{R}^d} \frac{p_{s}(\alpha)(0,y)p_{t-s}(\alpha)(y,0)}{p_{t}(\alpha)(0,0)} |y|^{\gamma} dy ds
\]

\[
= \int_{0}^{t/2} \int_{\mathbb{R}^d} \frac{p_{s}(\alpha)(0,y)p_{t-s}(\alpha)(y,0)}{p_{t}(\alpha)(0,0)} |y|^{\gamma} dy ds
\]

\[
+ \int_{t/2}^{t} \int_{\mathbb{R}^d} \frac{p_{s}(\alpha)(0,y)p_{t-s}(\alpha)(y,0)}{p_{t}(\alpha)(0,0)} |y|^{\gamma} dy ds
\]

\[
= 2 \int_{0}^{t/2} \int_{\mathbb{R}^d} \frac{p_{s}(\alpha)(0,y)p_{t-s}(\alpha)(y,0)}{p_{t}(\alpha)(0,0)} |y|^{\gamma} dy ds.
\]
Proof of Theorem 2

To estimate the right hand side we observe that for all $0 < s < t/2$ and all $y \in \mathbb{R}^d$,

$$p^{(\alpha)}_{t-s}(y,0) \leq p^{(\alpha)}_{t-s}(0,0) \leq p^{(\alpha)}_{t/2}(0,0).$$

By scaling

$$\frac{p^{(\alpha)}_{t/2}(0,0)}{p^{(\alpha)}_{t}(0,0)} = 2^{d/\alpha}$$

and therefore the right hand side is bounded above by

$$E_{0,0}^{t} \left( \int_{0}^{t} |X_{s}|^{\gamma} ds \right) \leq 2^{d/\alpha+1} \int_{0}^{t/2} \int_{\mathbb{R}^d} p^{(\alpha)}_{s}(0,y)|y|^{\gamma} dy ds$$

$$= 2^{d/\alpha+1} \int_{0}^{t/2} E^{0}(\|X_{s}\|^{\gamma}) ds$$

$$= 2^{d/\alpha+1} \int_{0}^{t/2} s^{\gamma/\alpha} E^{0}(\|X_{1}\|^{\gamma}) ds$$

$$= \frac{2^{d/\alpha+1}}{2^{\gamma/\alpha+1}} E^{0}(\|X_{1}\|^{\gamma}) \frac{t^{\gamma/\alpha+1}}{\gamma/\alpha + 1}.$$
Proof of Theorem 2

We now recall that $E^0(|X_1|^\gamma)$ is finite under our assumption that $\gamma < \alpha$. Thus we see that

$$E_{0,0}^t \left( \int_0^t |X_s|^\gamma ds \right) \leq C_{\alpha,\gamma,d} t^{\gamma/\alpha+1},$$

where the constant $C_{\alpha,\gamma,d}$ depends only on $\alpha$, $\gamma$ and $d$. We conclude that

$$\left| \int_{\mathbb{R}^d} E_{x,x}^t \left( \left[ \int_0^t V(X_s) ds \right]^2 \right) dx - t^2 \int_{\mathbb{R}^d} |V(x)|^2 dx \right| \leq 2tM\|V\|_1 E_{0,0}^t \left( \int_0^t |X_s|^\gamma ds \right) \leq M\|V\|_1 C_{\alpha,\gamma,d} t^{\gamma/\alpha+2}. $$
Proof of Theorem 2

Then
\[
\left| \frac{1}{p_t^\alpha(0)} \left( \text{Tr}(e^{-tH} - e^{-tH_0}) \right) + t \int_{\mathbb{R}^d} V(x) \, dx - \frac{1}{2} t^2 \int_{\mathbb{R}^d} |V(x)|^2 \, dx \right|
\leq C t^3 \| V \|_\infty e^t \| V \|_\infty \| V \|_1 + M \| V \|_1 C_{\alpha, \gamma, d} t^{\gamma/\alpha+2}
\leq C_{\alpha, \gamma, d} \| V \|_1 \left( \| V \|_\infty^2 e^t \| V \|_\infty t^3 + t^{\gamma/\alpha+2} \right).
\]

Rewriting this in the form stated in Theorem 2, we arrive at the announced bound
\[
\left| \left( \text{Tr}(e^{-tH} - e^{-tH_0}) \right) + p_t^\alpha(0) t \int_{\mathbb{R}^d} V(x) \, dx - p_t^{(\alpha)}(0) t^2 \frac{1}{2} \int_{\mathbb{R}^d} |V(x)|^2 \, dx \right|
\leq C_{\alpha, \gamma, d} \| V \|_1 p_t^{(\alpha)}(0) \left( \| V \|_\infty^2 e^t \| V \|_\infty t^3 + t^{\gamma/\alpha+2} \right),
\]
valid for all \( t > 0 \).
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\[ H_0^a = \Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}, \ a \geq 0, \ 0 < \beta < \alpha < 2 \]

* Taking \( 0 < \beta < \alpha < 2 \) and \( a \geq 0 \), consider the process \( Z_t^a = X_t + aY_t \), where \( X_t \) and \( Y_t \) are independent \( \alpha \)-stable and \( \beta \)-stable processes, respectively.

This process is called the independent sum of the symmetric \( \alpha \)-stable process \( X \) and the symmetric \( \beta \)-stable process \( Y \) with weight \( a \).

The infinitesimal generator of \( Z_t^a \) is \( \Delta^{\alpha/2} + a^\beta \Delta^{\beta/2} \). Acting on functions \( f \in C_0^\infty(\mathbb{R}^d) \) we have

\[
\left( \Delta^{\alpha/2} + a^\beta \Delta^{\beta/2} \right) f(x) = \\
\mathcal{A}_{d,-\alpha} \lim_{\epsilon \to 0^+} \int_{\{|y| > \epsilon\}} \left( \frac{1}{|y|^{d+\alpha}} + \frac{a^\beta}{|y|^{d+\beta}} \right) [f(x + y) - f(x)] dy,
\]

where \( \mathcal{A}_{d,-\alpha} \) is defined as before.
$H_0^a = \Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}, \ a \geq 0, \ 0 < \beta < \alpha < 2$

Taking $0 < \beta < \alpha < 2$ and $a \geq 0$, consider the process $Z_t^a = X_t + aY_t$, where $X_t$ and $Y_t$ are independent $\alpha$-stable and $\beta$-stable processes, respectively.

This process is called the independent sum of the symmetric $\alpha$-stable process $X$ and the symmetric $\beta$-stable process $Y$ with weight $a$.

The infinitesimal generator of $Z_t^a$ is $\Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$. Acting on functions $f \in C_0^\infty(\mathbb{R}^d)$ we have

\[
\left(\Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}\right)f(x) = \\
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\]

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Taking $0 < \beta < \alpha < 2$ and $a \geq 0$, consider the process 
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\[
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\]
\[
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\]
where $\mathcal{A}_{d,-\alpha}$ is defined as before.
\[ H_0^a = \Delta^{\alpha/2} + a^\beta \Delta^{\beta/2} \]

For the properties of the heat kernel (transition probabilities) for this operator see Chen and Kumagai (2008), Chen-Kim-Song (2012) or Jakubowski-Szczypkowski (2011), and references given there.

If we denote the heat kernel of this operator by \( p^a_t(x) \), we have that

\[
p^a_t(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} e^{-t(|\xi|^\alpha + a^\beta |\xi|^\beta)} d\xi = \int_0^\infty \frac{1}{(4\pi s)^{d/2}} e^{-\frac{|x|^2}{4s}} \eta^a_t(s) \, ds,
\]

where \( \eta^a_t(s) \) is be the density function of the sum of the \( \alpha/2 \)-stable subordinator and \( a^2 \)-times the \( \beta/2 \)-stable subordinator.

Again, this density is radial, symmetric, and decreasing in \( |x| \).
\[ H_0^a = \Delta^{\alpha/2} + a^\beta \Delta^{\beta/2} \]

For the properties of the heat kernel (transition probabilities) for this operator see Chen and Kumagai (2008), Chen-Kim-Song (2012) or Jakubowski-Szczypkowski (2011), and references given there.

If we denote the heat kernel of this operator by \( p_t^a(x) \), we have that

\[
p_t^a(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i x \cdot \xi} e^{-t(|\xi|^\alpha + a^\beta |\xi|^\beta)} \, d\xi = \int_0^\infty \frac{1}{(4\pi s)^{d/2}} e^{-\frac{|x|^2}{4s}} \eta_t^a(s) \, ds,
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$$H_0^a = \Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$$
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There is a constant \( C_{\alpha, \beta, d} \) such that for all \( x \in \mathbb{R}^d \) and \( t > 0 \),

\[
C_{\alpha, \beta, d}^{-1} f_t^a(x) \leq p_t^a(x) \leq C_{\alpha, \beta, d} f_t^a(x)
\]

where

\[
f_t^a(x) = \left( (a^\beta t)^{-d/\beta} \wedge t^{-d/\alpha} \right) \wedge \left( \frac{t}{|x|^{d+\alpha}} + \frac{a^\beta t}{|x|^{d+\beta}} \right).
\]
\[ H^a_0 = \Delta^{\alpha/2} + a^\beta \Delta^{\beta/2} \]

We note that for \( a = 0 \) this is just the estimate we obtained before. For any \( \gamma > 0 \) with \( 0 < \gamma < \beta < \alpha \) we have for any \( t > 0 \),

\[
\int_0^t E^0(|Z^a_s|^{\gamma}) ds \leq C_\gamma \left( \int_0^t E^0(|X_s|^{\gamma}) ds + a^\gamma \int_0^t E^0(|Y_s|^{\gamma}) ds \right)
\]

\[
= C_\gamma \left( E^0(|X_1|^{\gamma}) \frac{t^{\gamma/\alpha+1}}{\gamma/\alpha + 1} + a^\gamma E^0(|Y_1|^{\gamma}) \frac{t^{\gamma/\beta+1}}{\gamma/\beta + 1} \right)
\]

\[
= C_{a,\alpha,\beta,d} \left( \frac{t^{\gamma/\alpha+1}}{\gamma/\alpha + 1} + \frac{t^{\gamma/\beta+1}}{\gamma/\beta + 1} \right).
\]
Theorem (Bañuelos–Y.Y. (2012))

Let \( a \geq 0, 0 < \beta < \alpha < 2 \) and let \( H_0^a = \Delta^{\alpha/2} + a^\beta \Delta^{\beta/2} \). Suppose \( V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \) and that it is also uniformly Hölder continuous of order \( \gamma \), with \( 0 < \gamma < \beta \wedge 1 \), whenever \( 0 < \beta \leq 1 \), and with \( 0 < \gamma \leq 1 \), whenever \( 1 < \beta < 2 \). Let \( H^a = \Delta^{\alpha/2} + a^\beta \Delta^{\beta/2} + V \). Then for all \( t > 0 \),

\[
\left| \left( \text{Tr}(e^{-tH^a} - e^{-tH_0^a}) \right) + p_t^a(0) t \int_{\mathbb{R}^d} V(x) dx - p_t^a(0) \frac{1}{2} t^2 \int_{\mathbb{R}^d} |V(x)|^2 dx \right| 
\leq C_{a,\alpha,\beta,\gamma,d} \| V \|_1 p_t^a(0) \left( \| \| V \|_\infty^2 e^{t\| V \|_\infty} t^3 + t^{\gamma/\alpha+2} + t^{\gamma/\beta+2} \right),
\]

where the constant \( C_{a,\alpha,\beta,\gamma,d} \) depends only on \( a, \alpha, \beta, \gamma \) and \( d \). In particular, as \( t \downarrow 0 \),

\[
\text{Tr}(e^{-tH^a} - e^{-tH_0^a}) = p_t^a(0) \left( -t \int_{\mathbb{R}^d} V(x) dx + \frac{1}{2} t^2 \int_{\mathbb{R}^d} |V(x)|^2 dx + \mathcal{O}(t^{\gamma/\alpha+2}) \right).
\]
\(\alpha\)-stable relativistic process

This is again a Lévy process denote by \(X^m_t\) with characteristic function

\[
e^{-t\left((|\xi|^2+m^2/\alpha)^{\alpha/2}-m\right)} = E(e^{i\xi \cdot X^m_t}) = \int_{\mathbb{R}^d} e^{i\xi \cdot y} p_t^{(m,\alpha)}(y) dy,
\]

for any \(m \geq 0\) and \(0 < \alpha < 2\). As in the case of stable processes, \(X^m_t\) is a subordination of Brownian motion and in fact

\[
p_t^{(m,\alpha)}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} e^{-t\left((|\xi|^2+m^2/\alpha)^{\alpha/2}-m\right)} d\xi
\]

\[
= \int_0^\infty \frac{1}{(4\pi s)^{d/2}} e^{-\frac{|x|^2}{4s}} \eta_t^{m,\alpha}(s) ds,
\]

where \(\eta_t^{m,\alpha}(s)\) is the density of the subordinator with Bernstein function \(\Phi(\lambda) = (\lambda + m^2/\alpha)^{\alpha/2} - m\).
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$$e^{-t\left((|\xi|^2+m^2/\alpha)^{\alpha/2} - m\right)} = E(e^{i\xi \cdot X_t^m}) = \int_{\mathbb{R}^d} e^{i\xi \cdot y} p_t^{(m,\alpha)}(y) dy,$$

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$$= \int_0^\infty \frac{1}{(4\pi s)^{d/2}} e^{-\frac{|x|^2}{4s}} \eta_t^{m,\alpha}(s) ds,$$

where $\eta_t^{m,\alpha}(s)$ is the density of the subordinator with Bernstein function $\Phi(\lambda) = (\lambda + m^2/\alpha)^{\alpha/2} - m$. 
Relativistic Brownian motion

As before we see that \( p_t^{(m,\alpha)}(x) \) is radial, symmetric, and decreasing in \(|x|\).

- \( p_t^{(m,\alpha)}(x) = m^{d/\alpha} p_{mt}^{(1,\alpha)}(m^{1/\alpha} x) \).
- Grzywny- Ryznar - (2008)

\[
p_t^{(m,\alpha)}(x) = e^{mt} \int_0^\infty \frac{1}{(4\pi s)^{d/2}} e^{-|x|^2/4s} e^{-m^2/\alpha s} \eta_t^{\alpha/2}(s) \, ds,
\]

where \( \eta_t^{\alpha/2}(s) \) is the density for the \( \alpha/2 \)-stable subordinator. By scaling

\[
\eta_t^{\alpha/2}(s) = t^{-2/\alpha} \eta_1^{\alpha/2}(st^{-2/\alpha}).
\]

Hence changing variables leads to

\[
\lim_{t \downarrow 0} e^{-mt} t^{d/\alpha} p_t^{(m,\alpha)}(0) = \int_0^\infty \frac{1}{(4\pi s)^{d/2}} \eta_1^{\alpha/2}(s) \, ds = p_1^{\alpha}(0) = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha},
\]

where \( \omega_d \) is the volume of the unit ball in \( \mathbb{R}^d \).
**Relativistic Brownian motion**

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\[
p^{(m,\alpha)}_t(x) = m^{d/\alpha} p^{(1,\alpha)}_{mt}(m^{1/\alpha}x).
\]

Grzywny- Ryznar - (2008)

\[
p^{(m,\alpha)}_t(x) = e^{mt} \int_0^\infty \frac{1}{(4\pi s)^{d/2}} e^{-|x|^2/(4s)} e^{-m^2/\alpha s} \eta^{\alpha/2}_t(s) \, ds,
\]

where \( \eta^{\alpha/2}_t(s) \) is the density for the \( \alpha/2 \)-stable subordinator. By scaling

\[
\eta^{\alpha/2}_t(s) = t^{-2/\alpha} \eta^{\alpha/2}_1(st^{-2/\alpha}).
\]

Hence changing variables leads to

\[
\lim_{t \downarrow 0} e^{-mt} t^{d/\alpha} p^{(m,\alpha)}_t(0) = \int_0^\infty \frac{1}{(4\pi s)^{d/2}} \eta^{\alpha/2}_1(s) \, ds = p^{\alpha}_1(0) = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d}.
\]
Relativistic Brownian motion

The infinitesimal generator of $X_t^m$ is given by $m - (m^2/\alpha - \Delta)^{\alpha/2}$.

The case $\alpha = 1$ gives the generator $m - \sqrt{-\Delta + m^2}$ which is the free relativistic Hamiltonian. (see Carmona, Masters and Simon (1990))

Relativistic Brownian motion

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Relativistic Brownian motion

Chen, Kim and Song(2012) [Theorem 2.1]: For all $x \in \mathbb{R}^d$ and all $t \in (0, 1]$,

$$C_{\alpha,m,d}^{-1} t^{-d/\alpha} \wedge \frac{t \Psi(m^{\alpha}/|x|)}{|x|^{d+\alpha}} \leq p^{(m,\alpha)}_t(x) \leq C_{\alpha,m,d} t^{-d/\alpha} \wedge \frac{t \Psi(m^{\alpha}/|x|)}{|x|^{d+\alpha}},$$

where

$$\Psi(r) = 2^{-(d+\alpha)} \Gamma \left( \frac{d + \alpha}{2} \right)^{-1} \int_0^\infty s^{d+\alpha - 1} e^{-s/4} e^{-r^2/s} ds$$

which is a decreasing function of $r^2$ with $\Psi(0) = 1$, $\Psi(r) \leq 1$ and with

$$c_1^{-1} e^{-r} r^{(d+\alpha-1)/2} \leq \Psi(r) \leq c_1 e^{-r} r^{(d+\alpha-1)/2},$$

for all $r \geq 1$. 
Relativistic Brownian motion

Theorem (Bañuelos- Y.Y. (2012))

Let $H_0^m = m - \left(\frac{m^2}{\alpha} - \Delta\right)^{\alpha/2}$. Suppose $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ and that it is also uniformly Hölder continuous of order $\gamma$, with $0 < \gamma < \alpha \land 1$, whenever $0 < \alpha \leq 1$, and with $0 < \gamma \leq 1$, whenever $1 < \alpha < 2$. Let $H^m = m - \left(\frac{m^2}{\alpha} - \Delta\right)^{\alpha/2} + V$. Then for all $t > 0$,

$$\left| \left(\text{Tr}(e^{-tH^m} - e^{-tH_0^m})\right) + p_t^\alpha(0)t \int_{\mathbb{R}^d} V(x)dx - p_t^\alpha(0)\frac{1}{2}t^2 \int_{\mathbb{R}^d} |V(x)|^2dx \right| \leq C_{\alpha, \gamma, m, d} \|V\|_{1p}^{(m, \alpha)}(0) \left(\|V\|_\infty^2 e^t \|V\|_\infty t^3 + t^{\gamma/\alpha + 2}\right),$$

In particular,

$$\text{Tr}(e^{-tH^m} - e^{-tH_0^m}) = p_t^{(m, \alpha)}(0) \left(-t \int_{\mathbb{R}^d} V(x)dx + \frac{1}{2}t^2 \int_{\mathbb{R}^d} |V(x)|^2dx + O(t^{\gamma/\alpha + 2})\right),$$

as $t \downarrow 0$. 

Selma Yıldırım Yolcu
Heat Trace of Non-local Operators
Thank You!