# A study of stochastic equations by reducing them to ordinary differential equations.

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# Outline of the talk

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# Introduction

In this paper I will discuss an investigation of the qualitative properties of stochastic differential equations by reducing them to ordinary differential equations. Namely the stochastic case will be reduced to the deterministic case. My talk is dedicated to investigation of conditions under which such an approach is possible.

This paper consists of two parts. The first one deals with a study of an asymptotical behavior of solutions of stochastic systems for $t \to \infty$ by studying an asymptotical behavior of solutions of specific deterministic systems. In the second part I will talk about an existence of stable, invariant deterministic manifolds for stochastic systems, which allow to transform the original stochastic system into a deterministic system of ordinary differential equations. That is to investigate a stochastic system by reducing it to deterministic one.

# Statement of the problem

Consider the stochastic Ito system

$$dy = g(t, y)dt + \sigma(t, y)dW(t), \tag{1}$$

where

- $g(t, y), \sigma(t, y)$ are continuous in $t \geq 0, y \in \mathbb{R}^n$ and satisfy the global Lipschitz condition in $y$,
- $W(t)$ is a usual scalar Wiener process defined for $t \geq 0$ on the probability space $(\Omega, \mathbf{F}, P)$,
- $\{\mathcal{F}_t, t \geq 0\}$ is a family of $\sigma$-algebras s.t. $W(t)$ is consistent with $\mathcal{F}_t$.

The system (1) subject to the initial condition $y(t_0) = y_0$, $\mathbf{E}|y_0|^2 < \infty$ has a unique solution $y(t)$ for $t \geq t_0 \geq 0$.

# Statement of the problem

We study the asymptotic behavior of the solutions of (1) for $t \to \infty$. The analysis will be carried out using the well known in ordinary differential equations (ODE) method of *asymptotic equivalence*, when the solutions of the original system for $t \to \infty$ behave similarly to the solutions of a simpler system.

In our case we will be comparing the solutions of the original stochastic system with the solutions of a specially constructed deterministic system.

Along with (1) consider the deterministic system

$$dx = f(t, x)dt \tag{2}$$

# Statement of the problem

**Def.1.** *If for every solution* $y(t)$ *of (1) one can find a solution* $x(t)$ *of (2) s.t.*

$$\lim_{t\to\infty} \mathbf{E}|x(t) - y(t)|^2 = 0,$$

*then the system (2) is called asymptotically corresponding to the system (1) in square mean.*

**Def.2.** *If for every solution* $y(t)$ *of (1) one can find a solution* $x(t)$ *of (2) s.t., with probability one*

$$\lim_{t\to\infty} |x(t) - y(t)| = 0,$$

*then the system (2) is called asymptotically corresponding to the system (1) with probability 1.*

Our main question of interest is the following: under which conditions can one construct an ODE system (2) which is asymptotically corresponding to the stochastic system (1) in the sense of the Definitions 1 and 2?

# Existing results and motivation. Levinson THM

The approach described above is well known in ODE. The classic results in this direction are due to Wintner, Levinson and Yakubovich. Levinson Theorem (1948) gives the conditions of the asymptotic equivalence of linear systems

$$\frac{dy}{dt} = (A + B(t))y \tag{3}$$

$$\frac{dx}{dt} = Ax \tag{4}$$

**THM.**(Levinson) *If all solutions of (4) are bounded for* $t \geq 0$ *and the condition*

$$\int_0^\infty ||B(t)||dt < \infty, \tag{5}$$

*holds, then the systems (3) and (4) are asymptotically equivalent, i.e. one can find a one-to-one correspondence between their solutions* $y(t)$ *and* $x(t)$ *s.t.*

$$\lim_{t\to\infty} |x(t) - y(t)| = 0.$$

# Wintner THM

A result similar to Levinson Theorem was obtained by Wintner (1946). He considered the systems

$$\frac{dx}{dt} = A_1(t)x,$$
$$\frac{dy}{dt} = A_2(t)y.$$

**THM.**(Wintner) *Let* $A_1(t)$, $A_2(t)$ *be continuous for* $t \geq 0$ *and*

1. *every solution* $y(t)$ *is bounded for* $t \geq 0$;
2. $\liminf_{t\to\infty} \int\limits_0^t SpA_2(s)ds > -\infty$;
3. $\int\limits_0^\infty ||A_1(s) - A_2(s)||ds < \infty.$

*Then we can find a one-to-one correspondence between the solutions of these systems such that*

$$|x(t) - y(t)| = O(\int_t^\infty ||A_1(t) - A_2(t)||ds) \to 0, t \to \infty.$$

# Yakubovich THM

Both Levinson and Wintner theorems are based on one important condition: the solutions of unperturbed system are bounded for $t \to \infty$. Yakubovich (1951) generalized the results of Levinson and Wintner for the case of unbounded solutions of the unperturbed system. The following systems were considered

$$\frac{dy}{dt} = Ay + f(t, y) \tag{6}$$

$$\frac{dx}{dt} = Ax. \tag{7}$$

with $|f(t, y)| \leq g(t)|y|$ and satisfying the Lipschitz condition

$$|f(t, y_1) - f(t, y_2)| \leq g(t)|y_1 - y_2|,$$

$$f(t, 0) = 0.$$

# Yakubovich THM

**THM.**(Yakubovich) *Let $f$ satisfy the conditions above and*

$$\int_0^\infty t^{m+p-2} e^{\lambda t} g(t)dt < \infty.$$

*Then there is a one-to-one correspondence between the solutions of (6) and (7) s.t.*

$$|x(t) - y(t)| \to 0, t \to \infty, \tag{8}$$

*Moreover, the rate of convergence in (8) is specified.*
Here $\lambda = \max Re(\lambda_i)$, where $\{\lambda_i\}$ are the eigenvalues of $A$, $m$ is the maximal size of the Jordan cell for which $Re(\lambda_i) = \lambda$, $p$ is the maximal size of the Jordan cell for which $Re(\lambda_i) = 0$ , $p = 1$ if there are no such $\lambda_i$.

# Ahmetov THM

Among the recent results we mention the results of Ahmetov (2007). He studied the case $A = A(t)$. Consider the systems

$$\frac{dy}{dt} = (A(t) + B(t))y \tag{9}$$

and

$$\frac{dx}{dt} = A(t)x \tag{10}$$

Let $X(t)$ be the matriciant of the system (10), $X(0) = E$. Set $u := X^{-1}(t)y$ with $y$ solving (9). Then, clearly, $u$ satisfies

$$\dot{u} = P(t)u, \;\; P(t) = X^{-1}(t)B(t)X(t).$$

# Ahmetov THM

Consider the following conditions:

$C_1$) $\int\limits_0^\infty \|P(t)\| dt < \infty.$

$C_2$) $\lim\limits_{t\to\infty} X(t)\Phi(t) = 0$, where $\Phi(t)$ is the solution of

$$\dot{\phi} = P(t)(\phi + E) \tag{11}$$

s.t. $\Phi(t) \to 0, t \to \infty$ (the existence of such $\Phi$ was established).

**THM.**(Ahmetov) *Under the conditions $C_1$ and $C_2$ the systems (9) and (10) are asymptotically equivalent.*

# Motivation

After the pioneering works of Levinson, Wintner and Yakubovich, the problem of asymptotic equivalence of differential systems including linear, nonlinear and functional equations has been investigated by many authors.

Our goal is to establish similar results for stochastic systems, i.e. to compare the solutions of stochastic systems with the solutions of deterministic systems, and thus, to reduce the analysis of stochastic systems to the analysis of deterministic systems. This will enable us to address many issues of the qualitative theory of stochastic systems, including stability, dissipativity, dichotomy, theory of Lyapunov exponents etc., since those issues are well-established for deterministic systems.

In this work we will describe one of the applications of this method to the oscillation theory.

# Main results. Analog of Levinson Theorem

Consider two systems, one of which is deterministic and the other is stochastic:

$$dx = Axdt \tag{12}$$

$$dy = (A + B(t))ydt + D(t)y\,dW(t), \tag{13}$$

where $B(t)$ and $D(t)$ are measurable matrices for $t \geq 0$.

**THM.** *Let the system (12) be stable for $t \geq 0$, i.e. all its solutions are bounded for $t \geq 0$. If*

$$\int_0^\infty \|B(t)\|dt < \infty \text{ and } \int_0^\infty \|D(t)\|^2 dt < \infty,$$

*then the system (12) is asymptotically corresponding to the system (13) both in square mean and with probability 1.*

A similar result is also true for weakly nonlinear stochastic systems.

# Main results. More general setting

We now give a more general result. We consider the situation when the matrix of the linear part is nonconstant, and unperturbed system (12) may have unbounded solutions. This result is novel even for deterministic systems.
Consider nonlinear Ito system

$$dy = (A(t)y + f(t,y))dt + \sigma(t,y)dW(t) \tag{14}$$

and unperturbed system of ODEs

$$dx = A(t)xdt. \tag{15}$$

The matrix $A(t)$ is continuous and bounded on $\mathbb{R}^1$, $a := \sup_{t\in\mathbb{R}^1} \|A(t)\|$. The functions $f(t,y), \sigma(t,y)$ are continuous for $t \geq 0$, $y \in \mathbb{R}^n$ and Lipschitz in $y$. As we mentioned above, this guarantees the existence of the unique solution of $y(t_0) = y_0$, $\mathbf{E}|y_0|^2 < \infty$, for $t \geq t_0 \geq 0$.

Additionally, $f$ and $\sigma$ are small in the following sense: there exist nonnegative functions $\nu, \rho$ s.t. for $t \geq 0$ and $x \in \mathbb{R}^n$, $|f(t,x)| \leq \nu(t)|x|$, $|\sigma(t,x)| \leq \rho(t)|x|$.
Let $X(t)$ be the fundamental matrix of (15), $X(0) = E$.
Main condition on unperturbed system: (15) is exponentially dichotomic on $\mathbb{R}$, i.e. there exist positive constants $N_1, N_2, \nu_1, \nu_2$ and complementing projectors $P_1$, $P_2$ s.t.

$$\|X(t)P_1X^{-1}(s)\| \leq N_1 e^{-\nu_1(t-s)}, t \geq s$$

$$\|X(t)P_2X^{-1}(s)\| \leq N_2 e^{-\nu_2(t-s)}, s \geq t.$$

**THM.** *If* $\int\limits_0^\infty e^{at}\nu(t)dt < \infty$ *and* $\int\limits_0^\infty e^{2at}\rho^2(t)dt < \infty$, *then (15) is asymptotically corresponding to (14) in square mean. If the second condition is replaced with* $\int\limits_0^\infty te^{2at}\rho^2(t)dt < \infty$, *then (15) is asymptotically corresponding to (14) with probability 1.*

# Linear case

Consider the case when (14) is linear:

$$dy = (A(t) + B(t))ydt + D(t)ydW(t), \qquad (16)$$

where $B(t), D(t)$ are continuous deterministic matrices.

In this case we obtain a stronger result, namely, we can show that there exists a correspondence between the systems (16) and (15) s.t. each nontrivial solution of the system (16) corresponds to a nontrivial solution of (15). Here, nontrivial solution is the solution that becomes identically zero with zero probability.

**THM.** Let the system (15) be exponentially dichotomic on $\mathbb{R}$. Assume the following conditions hold:

1) $\int\limits_0^\infty e^{2at}\|B(t)\|^2 dt < \infty;$

2) $\int\limits_0^\infty te^{2at}\|D(t)\|^2 dt < \infty.$

Then the system (15) is asymptotically corresponding to the system (16) both in square mean and with probability one. Moreover, we can construct the correspondence between their solutions in such way that every nontrivial solution of the system (16) corresponds to a nontrivial solution of (15).

In the course of the proof of this Theorem, we obtain an important corollary about the behavior of solutions of (16).

**Corollary.**

- If the system (15) is exponentially stable, that is in the definition of dichotomy the projector $P_2$ is zero, all the solutions of (16) converge to zero for $t \to \infty$ both in square mean and with probability one.
- If (15) is exponentially unstable, that is in the definition of dichotomy the projector $P_1$ is zero, all nontrivial solutions of (16) go to infinity for $t \to \infty$ both in square mean and with probability one.

# Exponential dichotomy

Next we will talk about specific applications of these results. Earlier we have defined an exponential dichotomy for linear system of ordinary differential equations

$$dx = A(t)xdx. \tag{17}$$

**Def.3.** *We say that the system* (17) *is exponential dichotomous on the axis if the space $\mathbb{R}^n$ could be presented as the direct sum of two subspaces $\mathbb{R}^-, \mathbb{R}^+$: $\mathbb{R}^n = \mathbb{R}^- \oplus \mathbb{R}^+$ s.t. any solution $x(t, x_0)$ of the system* (17) *with $x(0, x_0) = x_0 \in \mathbb{R}^-$ satisfies the inequality*

$$|x(t, x_0)| \le K \exp -j(t-\tau)|x(\tau, x_0)|$$

*for $t \ge \tau$, and any solution $x(t, x_0)$ of the system* (17) *with $x(0, x_0) = x_0 \in \mathbb{R}^+$ satisfies the inequality*

$$|x(t, x_0)| \ge K_1 \exp j_1(t-\tau)|x(\tau, x_0)|$$

*for $t \ge \tau$, where $\tau \in \mathbb{R}$, some constants $K, K_1, j, j_1$ are undependable of $\tau$ and $x_0$.* That is the solution which has a start from subspace $\mathbb{R}^-$ ( $\mathbb{R}^+$) /stable (unstable) subspace/ decrease (increase) exponentially.

Similar problems for stochastic systems

$$dy = A(t)ydy + B(t)ydW(t) \tag{18}$$

were studied in collaboration with A.M.Samojlenko.

**Def.4.** *We say that the system* (18) *is exponential dichotomous in square mean for* $t \geq 0$ *if the space* $\mathbb{R}^n$ *could be presented as the direct sum of two subspaces* $\mathbb{R}^-, \mathbb{R}^+$ *s.t. any solution* $x(t, x_0)$ *of the system* (18) *with* $x(0, x_0) = x_0 \in \mathbb{R}^-$ *satisfies the inequality*

$$\mathbb{E}|x(t, x_0)|^2 \leq K \exp -j(t-\tau)\mathbb{E}|x(\tau, x_0)|^2 \tag{19}$$

*for* $t \geq \tau \geq 0$, *and any solution* $x(t, x_0)$ *of the system* (18) *with* $x(0, x_0) = x_0 \in \mathbb{R}^+$ *satisfies the inequality*

$$\mathbb{E}|x(t, x_0)|^2 \geq K_1 \exp j_1(t-\tau)\mathbb{E}|x(\tau, x_0)|^2 \tag{20}$$

*for* $t \geq \tau \geq 0$, *where* $\tau \in \mathbb{R}$, *some constants* $K, K_1, j, j_1$ *are undependable of* $\tau$ *and* $x_0$. That is the second mathematical moments of solutions decrease or increase exponentially, which depends on a subspace where the solutions have a start.

Next we are interested with the behavior of trajectories with probability 1. Obtained above results about asymptotical equivalence give us an opportunity to study this issue.

**THM.** *Let the matrices* $B(t)$ *and* $D(t)$ *from the system* (16) *tend to zero as* $t \to \infty$ *and conditions of the previous theorem are satisfied. Then* $\exists t_0 > 0$ *s.t. for* $t \geq t_0$ *the system* (16) *is exponentially dichotomous. Moreover solutions which have a start from* $\mathbb{R}^-$ *(from Def.4) tend to zero with probability 1 as* $t \to \infty$, *and solutions which do not have a start from* $\mathbb{R}^-$ *tend over norm to infinity with probability 1 as* $t \to \infty$.

# Application to a study of the Oscillation Theory for stochastic linear second order equations

We now illustrate the application of the theory above to the practical example. Based on the theorem on asymptotic correspondence we built the analog of Sturm oscillation theory for linear stochastic second order equations of the type

$$\ddot{x} + (p(t) + q(t)\dot{W}(t))x = 0, \tag{21}$$

where $x \in \mathbb{R}$, $t \geq 0$; $p(t), q(t)$ are continuous functions and $\dot{W}(t)$ is the generalized derivative of the standard Wiener process.
The equation (21) is a mathematical model of various real-life processes in mechanics, which are influenced by random factors.
The corresponding unperturbed ODE

$$\ddot{x} + p(t)x = 0 \tag{22}$$

describes the motion of a mechanical system, which is influenced by the elastic force with elasticity coefficient $p(t)$.

# Application to the Oscillations Theory

However, the coefficient $p(t)$ is only the average value of the true elasticity coefficient. Its true value is a random process with a small correlation interval. Therefore the model (21) is more accurate.

We now set aside the mechanical aspect and focus on the mathematical model. The equation (21) is not rigorous as is because the derivative of Wiener process does not exist. Therefore we will understand (21) as a system of stochastic Ito equations

$$\begin{cases} dx_1 = x_2 dt \\ dx_2 = -p(t)x_1 dt - q(t)x_1 dW(t). \end{cases} \tag{23}$$

which is absolutely rigorous now. In this notation, $x(t) = x_1(t)$.

We now introduce the notion of the first zero of the solution of (23), then introduce the concepts of oscillating / non-oscillating solutions on semi-axis.

Since the solutions of (23) are random processes with certain properties, introducing the concept of zero requires a subtle construction, unlike in the deterministic case.

Clearly the zeros of the solution are the random variables. Therefore, we need to introduce a zero in such a way that it is a Markov's moment relatively to the family of $\sigma$- algebras in the definition of the solution.

Define the random variable $\tau_1$ as

$$\tau_1 := \inf\{t > t_0 | x_1(t) = 0\},$$

if the set over which infimum is taken in non-empty and $\tau_1 = +\infty$ otherwise.

**Def.** *The random variable $\tau_1$ is called the first zero of $x(t)$ on the interval $t > t_0$ if $\tau_1 < +\infty$ with probability 1.*

Since (23) is linear and $x_1(t)$ is smooth, it is not difficult to show that in some neighborhood of the first zero $\tau_1$ the component $x_1(t)$ is different from zero. Thus we can define the random variable $\tau_2$ as

$$\tau_2 := \inf\{t > \tau_1 : x_1(t) = 0\},$$

if the set over which infimum is taken in non-empty and $\tau_2 = +\infty$ otherwise. If $\tau_2 < +\infty$ with probability 1, it is called *the second zero* of $x(t)$ on the interval $t > t_0$.

By induction we can define a sequence of zeros $\{\tau_n\}$ of the solution $x(t)$ on the interval $t > t_0$. If $t_0 = 0$, then we have zeros on the semi-axis $t \geq 0$.

This sequence of zeros is a sequence of Markov's moments, which often enables us to work with them as with deterministic (for example, we can perform stochastic differentiation on the interval $(\tau_{n-1}, \tau_n)$.)

**Def.** *A nontrivial solution $x(t)$ of (21) is called oscillating on the semi-axis $t > 0$ if it has infinitely many zeros on this interval.*

Let $I = (t_0, t_1)$ be a bounded interval, $x(t)$ be a nontrivial solution of (21) for $t \geq t_0$, $\tau_1$ is the first zero on the interval $t > t_0$ (assuming that it exists).

**Def.** *We say that the solution $x(t)$ has the first zero on the interval $I$ if $t_0 < \tau_1 < t_1$ with probability 1.*

**Def.** *The solution $x(t)$ is called oscillating on $I$ if it has at least two zeros $\tau_1, \tau_2$ on the interval $t > t_0$ and with probability 1, $\tau_1 \in I$ and $\tau_2 \in I$.*

The following analog of the theorem from the classic Sturm oscillation theory holds:

**Lemma.** *With probability 1 any nontrivial solution of (21) has at most finite number of zeros on a finite interval.*

We now give the analog of the Comparison Theorem. Along with (21) consider a similar equation

$$\ddot{y} + (\tilde{p}(t) + q(t)\dot{W}(t))y = 0. \tag{24}$$

From now on, $I = (t_0, t_1)$ is a bounded interval.

**THM.**(Comparison) *Assume $\tilde{p}(t) \geq p(t)$ on $I$. Then, if $\tau_1, \tau_2$ are two consecutive zeros of a solution of (21) on $I$, then any solution $y(t)$ of the equation (23) has at least one zero $\tau$ on the interval $I$ and, with probability 1, $\tau_1 \leq \tau \leq \tau_2$.*

As a corollary of the Comparison Theorem, we get

**THM.** *Let* $\tau_1, \tau_2$ *be two consecutive zeros of a solution* $x(t)$ *of (21) on* $I$. *Then any other solution* $\tilde{x}(t)$ *of (21), which is linearly independent with* $x(t)$, *has exactly one zero on the open stochastic interval* $(\tau_1, \tau_2)$.

**THM.**(Non-oscillation) *If* $p(t) \leq 0$ *for* $t \geq t_0$ *and* $\int\limits_{t_0}^{\infty} q^2(t)dt < \infty$, *then all nontrivial solutions of (21) are not oscillating on the semi-axis.*

# Auxiliary equation

Consider the equation

$$\ddot{x} + (a^2 + q(t)\dot{W}(t))x = 0, \qquad (25)$$

which plays the role of a standard equation

$$\ddot{x} + a^2 x = 0$$

in the deterministic oscillation theory. Since we can solve the last equation, we know everything about its solutions, e.g. we know that the exact distance between two consecutive zeros is $\frac{\pi}{a}$. Unfortunately, its stochastic analog (25) cannot be solved explicitly. However, we can obtain some asymptotic results for it.

**THM.** *Let* $\int\limits_0^\infty q^2(t)dt < \infty$. *Then all the solutions* $x(t)$ *of the equation (24) are oscillating on the semi-axis and with probability 1*

$$\xi_n = \tau_{n+1} - \tau_n \to \frac{\pi}{a}, n \to \infty,$$

*where* $\tau_n$ *and* $\tau_{n+1}$ *are two consecutive zeros.*

Denote

$$m = \inf_{t\geq 0} p(t) \text{ and } M = \sup_{t\geq 0} p(t).$$

**THM.** *Let $p(t)$ be a continuous function and such that on any subinterval of the positive semiaxis the condition $0 < m < p(t) < M < \infty$ holds. If, additionally,*

$$\int_0^\infty q^2(t)dt < \infty,$$

*then all the solutions of (21) are oscillating on the semiaxis $[0, \infty)$. Moreover, for any $\varepsilon > 0$ s.t. $\varepsilon < \frac{2}{\sqrt{M}}$ there exists a random variable $T(\omega) > 0$, so that for any nontrivial solution $x(t)$ of the equation (21) for $\tau_n \geq T(\omega)$ we can estimate the distance between two consecutive zeros with probability 1:*

$$\frac{\pi}{\sqrt{M}} - \varepsilon \leq \tau_{n+1} - \tau_n \leq \frac{\pi}{\sqrt{m}} + \varepsilon. \tag{26}$$

**Example 1.** Bessel equation perturbed with white noise:

$$\ddot{x}(t) + (1 - \frac{\nu^2 - \frac{1}{4}}{t^2} + q(t)\dot{w}(t))x(t) = 0, \qquad (27)$$

where $\int\limits_0^\infty q^2(t)dt < \infty$. The previous theorem implies that with probability 1 we can estimate the distance between two consecutive zeros of (27): $\lim\limits_{n\to\infty}(\tau_{n+1} - \tau_n) = \pi$. This result is the same as for the ordinary equations.

**Example 2.** Airy equation perturbed with white noise:

$$\ddot{x}(t) + (t + q(t)\dot{w}(t))x(t) = 0 \qquad (28)$$

where, as before, $\int\limits_0^\infty q^2(t)dt < \infty$. From the previous theorem, $\lim\limits_{n\to\infty}(\tau_{n+1} - \tau_n) = 0$.

We consider the stochastic system

$$dx = a(t,x)dt + \sum_{r=1}^{k} b_r(t,x)dW_r(t), \qquad (29)$$

where $t \geq 0$, $x \in \mathbb{R}^n$, $a$, $b_r$, $r = \overline{1,k}$ are vectors in $\mathbb{R}^n$, $W_1, ..., W_r$ are undependable, scalar standard Wiener processes.
We assume that $a$, $b_r$ are non random and s.t. the equation (29) has unique strong solution with initial values $x(t_0) = x_0 \in \mathbb{R}^n$ for $t \geq t_0$.

We denote $S$ is some Borel set in $\{t \geq 0\} \times \mathbb{R}^n$. Also let $S_t$ be a set in $\mathbb{R}^n$ s.t. $S_t = \{x \in \mathbb{R}^n : (t, x) \in S\}$ and let $S_t \neq \varnothing$ for all $t \geq 0$.

**Def.5.** *We say that a set $S$ is positively invariant for the system* (29) *if*

$$P\{(t, x(t, t_0, x_0)) \in S, \forall t \geq t_0\} = 1 \tag{30}$$

*for $(t_0, x_o(\omega)) \in S$ with probability 1, where $x(t, t_0, x_0)$ is a solution of the system* (29) *s.t. $x(t_0, t_0, x_0) = x_0$ for $t_0 \geq 0$*

Let us consider the case when over an invariant set the original stochastic system could be transformed in deterministic one. Then an investigation of stochastic system stability could be reduced to an investigation of deterministic system stability.
We consider for $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $t \geq 0$ the Ito type stochastic system:

$$\begin{cases} dx = X(x, y)dt \\ dy = A(t)ydy + \sigma(t, x, y)dW(t) \end{cases} \tag{31}$$

where $\sigma(t, x, y)$ is $m \times r$-dimensional matrix, $W(t)$ is $r$-dimensional Wiener process with undependable components.
We believe $X(x, y)$, $\sigma(t, x, y)$ are Lipshitz over $x, y$ for $t \geq 0$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ with constants $L_1$ and $L_2$ correspondingly.

# Invariant sets of stochastic systems

Let the matriciant $\Phi(t, \tau)$ of the linear system

$$\frac{dy}{dx} = A(t)y$$

permits an estimation

$$\|\Phi(t, \tau)\| \leq Re^{-\rho(t-\tau)}, \tag{32}$$

where $R, \rho$ are positive constants, undependable of $t$ and $\tau$.
Also we suppose that

$$\sigma(t, x, 0) \equiv 0 \tag{33}$$

The last equivalence means that the system (31) has the invariant set $\{y = 0\}$, over which this systemm could be transformed into deterministic one

$$\frac{dx}{dt} = X(x, 0) \tag{34}$$

As well let $X(0, 0) = 0$.

Hence the stochastic system (31) has zero solution $x = 0, y = 0$, so we will investigate the stability of this solution.
For convenience we define $z = (x, y)$, $z_0 = (x_0, y_0)$. The stability we understand in the sense of square mean, that is in the sense of the next definition.
**Def.6.** *We say that zero solution of the system* (31) *is square mean stable if for all* $\varepsilon > 0$ *exists* $\delta$ *s.t.*

$$\mathbb{E}|z(t, t_0, z_0)|^2 < \varepsilon,$$

*for* $t \geq 0$ *and* $\mathbb{E}|z_0|^2 < \delta$.

**THM.** *Let the zero solution of the system* (34) *be uniformly asymptotically stable and*

$$L_2 < \frac{\sqrt{2\rho}}{R}.$$

*Then the zero solution of the system* (31) *is square mean stable uniformly on* $t_0 \geq 0$.

I would like to emphasize that this paper introduced only one of the results of such type, when an investigation of stochastic system stability was reduced to an investigation of deterministic system stability.

Analogous results could be obtained for other type systems. Similar methods could be used to study the stability of more complex invariant sets than point ones. In this case there is need to use Lyapunov functions.

## Thank you for your attention!!!

Announcement: conference in differential and stochastic differential equations, in honor of 75th anniversary of A.M. Samoilenko
**June 23-30, 2012, Sevastopol, Ukraine**
More information: **www.imath.org.ua**