The Contact Process on the Complete Graph with Random, Vertex-dependent, Infection Rates

Jonathon Peterson

Purdue University
Department of Mathematics

December 2, 2011
Contact Process

Graph $G = (V, E)$.
Markov Process $\eta_t$ on $\{0, 1\}^V$.

$\eta_t(x) = 0$ healthy
$\eta_t(x) = 1$ infected.

Contact Process dynamics:

<table>
<thead>
<tr>
<th>Transition</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Infected $\rightarrow$ Healthy</td>
<td>$1$</td>
</tr>
<tr>
<td>Healthy $\rightarrow$ Infected</td>
<td>$\lambda$ (# infected neighbors)</td>
</tr>
</tbody>
</table>

Classical examples: $\mathbb{Z}^d$ or the homogeneous tree $T_d$. 
Phase Transition

Q: How does the infection rate \( \lambda \) affect the long term behavior of \( \eta_t \)?

\[
\tau = \inf\{ t \geq 0 : \eta_t \equiv 0 \} \quad \text{(lifetime of infection)}.
\]

\[
\lambda_c(G) = \sup\{ \lambda \geq 0 : P^{\{x\}}(\tau < \infty) = 1 \}
\]
Phase Transition

Q: How does the infection rate $\lambda$ affect the long term behavior of $\eta_t$?

$$\tau = \inf\{ t \geq 0 : \eta_t \equiv 0 \} \quad \text{ (lifetime of infection)}.$$ 

$$\lambda_c(G) = \sup\{ \lambda \geq 0 : P^x(\tau < \infty) = 1 \}$$

Finite Graphs: Contact process on $\{1, 2, \ldots, n\}^d = G_n$
Phase Transition

Q: How does the infection rate $\lambda$ affect the long term behavior of $\eta_t$?

$$\tau = \inf\{t \geq 0 : \eta_t \equiv 0\} \quad \text{(lifetime of infection)}.$$

$$\lambda_c(G) = \sup\{\lambda \geq 0 : P^x(\tau < \infty) = 1\}$$

Finite Graphs: Contact process on $\{1, 2, \ldots, n\}^d = G_n$

$$\lambda < \lambda_c(\mathbb{Z}^d) \implies P^{G_n}(\tau < C \log n) \to 1$$

$$\lambda > \lambda_c(\mathbb{Z}^d) \implies P^{G_n}(\tau > e^{cn^d}) \to 1.$$
Power-law Random Graphs

Limiting degree distribution $P(D_0 \geq k) \sim Ck^{-(\alpha-1)}$. 

$\rho = \frac{1}{\sum_i w_i}$
Power-law Random Graphs

Limiting degree distribution \( P(D_0 \geq k) \sim Ck^{-(\alpha-1)} \).

- **Barbasi-Albert**: Preferential attachment
Power-law Random Graphs

Limiting degree distribution \( P(D_0 \geq k) \sim Ck^{-(\alpha-1)} \).

- **Barbasi-Albert**: Preferential attachment
- **Newman-Strogatz-Watts**: Configuration model
Power-law Random Graphs

Limiting degree distribution $P(D_0 \geq k) \sim Ck^{-(\alpha-1)}$.

- **Barbasi-Albert**: Preferential attachment
- **Newman-Strogatz-Watts**: Configuration model
- **Chung-Lu**: Vertex weights $w_i$ (expected degree).

\[ P(i \leftrightarrow j) = \rho w_i w_j \quad (\rho = 1 / \sum_i w_i) \]

\[ \approx \frac{w_i w_j}{nE[w_1]} \, . \]
Contact Process on Power-law Random Graphs

$G_n$ a power-law random graph (index $\alpha$). Contact process on $G_n$ with infection rate $\lambda$.

**Physics:** Mean field treatment (non-rigorous)

- $\lambda_c = 0$ if $\alpha \leq 3$
- $\lambda_c > 0$ if $\alpha > 3$
Contact Process on Power-law Random Graphs

$G_n$ a power-law random graph (index $\alpha$). Contact process on $G_n$ with infection rate $\lambda$.

**Physics:** Mean field treatment (non-rigorous)
- $\lambda_c = 0$ if $\alpha \leq 3$
- $\lambda_c > 0$ if $\alpha > 3$

**Mathematics:** $\lambda_c = 0$ for any $\alpha \geq 3$.
- Berger, Borgs, J. Chayes, & Saberi - Preferential attachment
- Shirshendu Chatterjee & Durrett - Configuration model
Contact Process with Vertex-dependent Edge Weights

Motivation:

- Sex in Sweden: Power law random graph
- Spread of STD: Contact process
Contact Process with Vertex-dependent Edge Weights

**Motivation:**
- Sex in Sweden: Power law random graph (past contacts)
- Spread of STD: Contact process (future contacts)
Contact Process with Vertex-dependent Edge Weights

Motivation:

- Sex in Sweden: Power law random graph (past contacts)
- Spread of STD: Contact process (future contacts)

Model:

- Assign vertices i.i.d. weights $w_i$ with distribution $\mu$.
- Infection rate between $i$ and $j$: $\frac{\lambda w_i w_j}{n}$.
Contact Process with Vertex-dependent Edge Weights

Motivation:
- Sex in Sweden: Power law random graph (past contacts)
- Spread of STD: Contact process (future contacts)

Model:
- Assign vertices i.i.d. weights $w_i$ with distribution $\mu$.
- Infection rate between $i$ and $j$: $\frac{\lambda w_i w_j}{n}$.

If $\mu(w_i > x) \sim Cx^{-\alpha}$, graph of potential transmissions by time $t$ is a PLRG.
Mean field computation: $\lambda_c = 1 / E_\mu [w_1^2]$

**Theorem (P. ’11)**

- If $\lambda < 1 / E_\mu [w_1^2]$, there exists $C > 0$ such that
  $$P^{[n]}_w(\tau \leq C \log n) \to 1.$$

- If $\lambda > 1 / E_\mu [w_1^2]$, there exists $c > 0$ such that
  $$P^{[n]}_w(\tau \geq e^{cn}) \to 1.$$
**Results - Quasi-stationary distribution**

**Mean field computation:** Quasi-stationary distribution

\[ P_w^n(\eta_t(i) = 1) \approx \frac{\sigma(\lambda)w_i}{1 + \sigma(\lambda)w_i}, \quad \text{where } 1 = \lambda E\mu \left[ \frac{w_1^2}{1 + \sigma(\lambda)w_1} \right] \]

**Theorem (P. ’11)**

If \( \lambda > \lambda_c \), then for any \( \varepsilon > 0 \) there exists \( c, C > 0 \) such that

\[
\lim_{n \to \infty} \sup_{C \log n \leq t \leq e^{cn}} \sup_{i \leq n} \left| P_w^n(\eta_t(i) = 1) - \frac{\sigma(\lambda)w_i}{1 + \sigma(\lambda)w_i} \right| \leq \varepsilon.
\]