The Contact Process on the Complete Graph with Random, Vertex-dependent, Infection Rates

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Contact Process

Graph $G = (V, E)$.
Markov Process η_t on $\{0, 1\}^V$.

$$\eta_t(x) = 0 \text{ healthy}$$
$$\eta_t(x) = 1 \text{ infected .}$$

Contact Process dynamics:

Transition	**rate**
Infected $\to$ Healthy	1
Healty $\to$ Infected	λ (# infected neighbors).

Classical examples: $\mathbb{Z}^d$ or the homogeneous tree $\mathbf{T}_d$.

Phase Transition

Q: How does the infection rate λ affect the long term behavior of η_t?

$$\tau = \inf\{t \geq 0 : \eta_t \equiv 0\} \quad \text{(lifetime of infection).}$$

$$\lambda_c(G) = \sup\{\lambda \geq 0 : P^{\{x\}}(\tau < \infty) = 1\}$$

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$$\lambda < \lambda_c(\mathbb{Z}^d) \quad \Longrightarrow \quad P^{G_n}(\tau < C \log n) \to 1$$
$$\lambda > \lambda_c(\mathbb{Z}^d) \quad \Longrightarrow \quad P^{G_n}(\tau > e^{cn^d}) \to 1.$$

Power-law Random Graphs

Limiting degree distribution $P(D_0 \geq k) \sim Ck^{-(\alpha-1)}$.

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- **Barbasi-Albert:** Preferential attachment
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- **Chung-Lu:** Vertex weights w_i (expected degree).

$$P(i \leftrightarrow j) = \rho w_i w_j \qquad (\rho = 1/\sum_i w_i)$$
$$\approx \frac{w_i w_j}{nE[w_1]}.$$

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G_n a power-law random graph (index α).
Contact process on G_n with infection rate λ.

Physics: Mean field treatment (non-rigorous)

- $\lambda_c = 0$ if $\alpha \leq 3$
- $\lambda_c > 0$ if $\alpha > 3$

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- $\lambda_c > 0$ if $\alpha > 3$

Mathematics: $\lambda_c = 0$ for any $\alpha \geq 3$.

- Berger, Borgs, J. Chayes, & Saberi - Preferential attachment
- Shirshendu Chatterjee & Durrett - Configuration model

Contact Process with Vertex-dependent Edge Weights

Motivation:

- Sex in Sweden: Power law random graph
- Spread of STD: Contact process

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- Assign vertices i.i.d. weights w_i with distribution μ.
- Infection rate between i and j: $\frac{\lambda w_i w_j}{n}$.

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If $\mu(w_i > x) \sim Cx^{-\alpha}$, graph of *potential transmissions* by time t is a PLRG.

Results - Phase Transition

Mean field computation: $\lambda_c = 1/E_\mu[w_1^2]$

Theorem (P. '11)

- *If* $\lambda < 1/E_\mu[w_1^2]$, *there exists* $C > 0$ *such that*

$$P_w^{[n]}(\tau \leq C \log n) \to 1.$$

- *If* $\lambda > 1/E_\mu[w_1^2]$, *there exists* $c > 0$ *such that*

$$P_w^{[n]}(\tau \geq e^{cn}) \to 1.$$

Results - Quasi-stationary distribution

Mean field computation: Quasi-stationary distribution

$$P_w^{[n]}(\eta_t(i) = 1) \approx \frac{\sigma(\lambda) w_i}{1 + \sigma(\lambda) w_i}, \quad \text{where } 1 = \lambda E_\mu \left[\frac{w_1^2}{1 + \sigma(\lambda) w_1} \right]$$

Theorem (P. '11)

If $\lambda > \lambda_c$, then for any $\varepsilon > 0$ there exists $c, C > 0$ such that

$$\lim_{n \to \infty} \sup_{C \log n \le t \le e^{cn}} \sup_{i \le n} \left| P_w^{[n]}(\eta_t(i) = 1) - \frac{\sigma(\lambda) w_i}{1 + \sigma(\lambda) w_i} \right| \le \varepsilon.$$