Self-interacting Random Walks

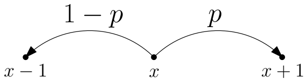
Jonathon Peterson

Purdue University
Department of Mathematics

September 16, 2013

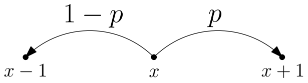
Simple Random Walk

Simple random walk on $\mathbb{Z}$

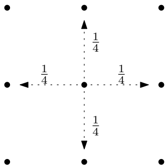


Simple Random Walk

Simple random walk on $\mathbb{Z}$



Simple (symmetric) random walk on $\mathbb{Z}^d$



Recurrence and Transience

Recurrent: RW returns to starting location infinitely many times.
Transient: RW returns only finitely many times.

Recurrence and Transience

Recurrent: RW returns to starting location infinitely many times.
Transient: RW returns only finitely many times.

Theorem

Simple RW on $\mathbb{Z}$ is recurrent $\iff p = 1/2$.

Recurrence and Transience

Recurrent: RW returns to starting location infinitely many times.
Transient: RW returns only finitely many times.

Theorem

Simple RW on $\mathbb{Z}$ is recurrent $\iff p = 1/2$.

Theorem

Simple symmetric RW on $\mathbb{Z}^d$ is

- *recurrent if $d = 1, 2$.*
- *transient if $d \geq 3$.*

Recurrence and Transience

Recurrent: RW returns to starting location infinitely many times.
Transient: RW returns only finitely many times.

Theorem

Simple RW on $\mathbb{Z}$ is recurrent $\iff$ $p = 1/2$.

Theorem

Simple symmetric RW on $\mathbb{Z}^d$ is

- *recurrent if $d = 1, 2$.*
- *transient if $d \geq 3$.*

Random walks and electric networks by Doyle and Snell.
`http://arxiv.org/abs/math/0001057`

Simple Random Walk

Theorem (Law of Large Numbers)

$$\lim_{n\to\infty} \frac{S_n}{n} = 2p - 1, \quad P - a.s.$$

Proof:

$$S_n = \sum_{i=1}^{n} \xi_i$$

Simple Random Walk

Theorem (Law of Large Numbers)

$$\lim_{n\to\infty} \frac{S_n}{n} = 2p-1, \quad P-a.s.$$

Proof:

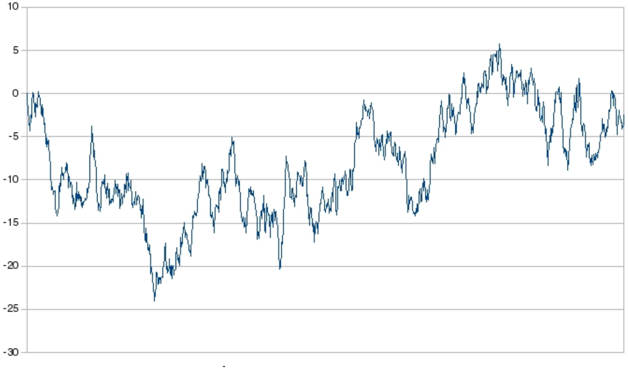
$$S_n = \sum_{i=1}^{n} \xi_i$$

Theorem (Central Limit Theorem)

Let $v = 2p-1$ *and* $\sigma = 2\sqrt{p(1-p)}$. *Then,*

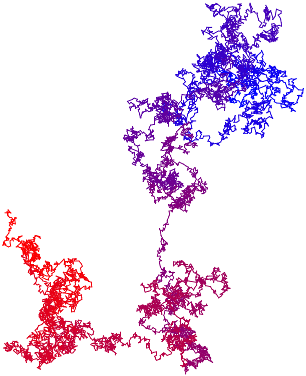
$$\lim_{n\to\infty} P\left(\frac{S_n - nv}{\sigma\sqrt{n}} \le x\right) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt.$$

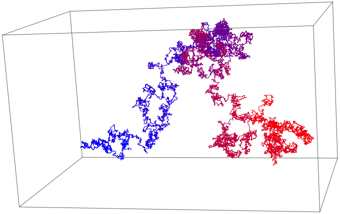
Brownian Motion



$$t \mapsto \frac{S_{\lfloor nt \rfloor} - ntv}{\sigma\sqrt{n}}$$

Brownian Motion (higher dimensions)





Classical random walks

$$S_n = \sum_{i=1}^{n} \xi_i$$

with $\{\xi_i\}_{i\geq 1}$ independent and identically distributed.

Classical random walks

$$S_n = \sum_{i=1}^{n} \xi_i$$

with $\{\xi_i\}_{i\geq 1}$ independent and identically distributed.

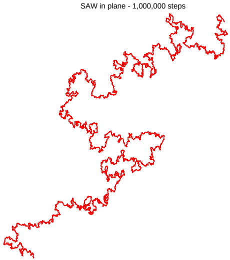
Drawbacks of the classical model

- Spatial homogeneity
- Temporal homogeneity
- Independent increments

Self-avoiding random walk

$\{X_k\}_{k \leq n}$ - uniform from all self-avoiding paths of length n.

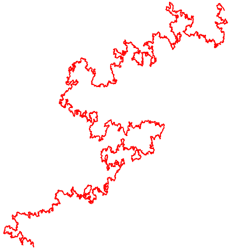
Simulation from Tom Kennedy



Self-avoiding random walk

$\{X_k\}_{k \leq n}$ - uniform from all self-avoiding paths of length n.

Simulation from Tom Kennedy

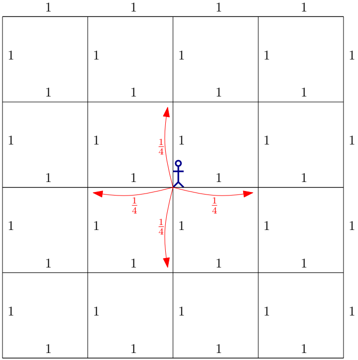


Conjecture

The paths $\{X_k / n^{2/3}\}_{k \leq n}$ converge in distribution to an $SLE_{8/3}$ process.

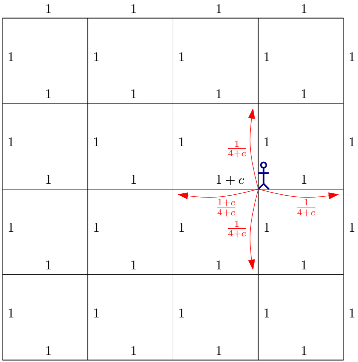
Reinforced random walk

- Initial edge weights $\equiv 1$
- Edge weights increase by c when crossed
- Steps taken proportional to edge weights



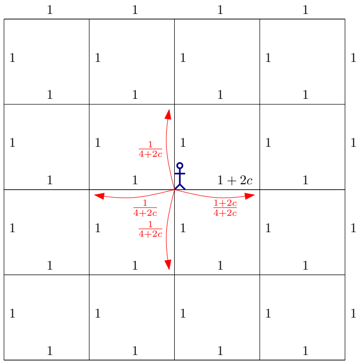
Reinforced random walk

- Initial edge weights $\equiv 1$
- Edge weights increase by c when crossed
- Steps taken proportional to edge weights



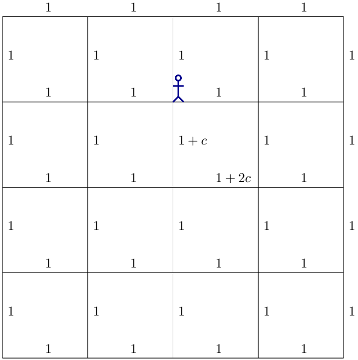
Reinforced random walk

- Initial edge weights $\equiv 1$
- Edge weights increase by c when crossed
- Steps taken proportional to edge weights



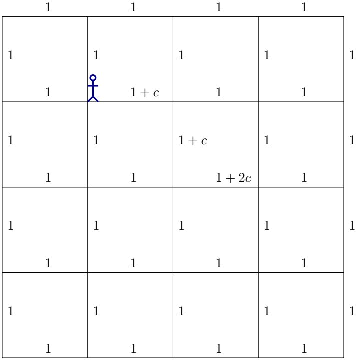
Reinforced random walk

- Initial edge weights $\equiv 1$
- Edge weights increase by c when crossed
- Steps taken proportional to edge weights



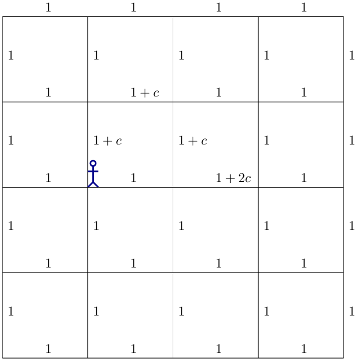
Reinforced random walk

- Initial edge weights $\equiv 1$
- Edge weights increase by c when crossed
- Steps taken proportional to edge weights



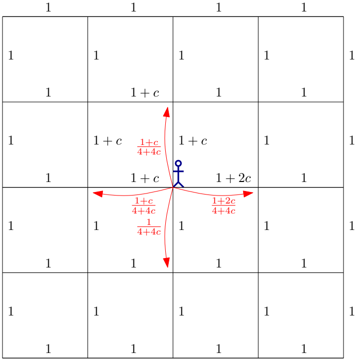
Reinforced random walk

- Initial edge weights $\equiv 1$
- Edge weights increase by c when crossed
- Steps taken proportional to edge weights



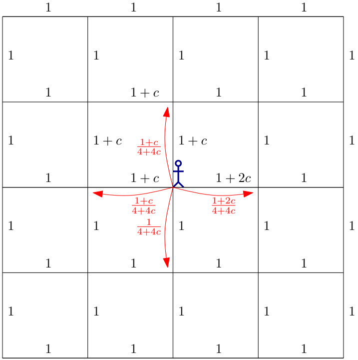
Reinforced random walk

- Initial edge weights $\equiv 1$
- Edge weights increase by c when crossed
- Steps taken proportional to edge weights



Reinforced random walk

- Initial edge weights $\equiv 1$
- Edge weights increase by c when crossed
- Steps taken proportional to edge weights

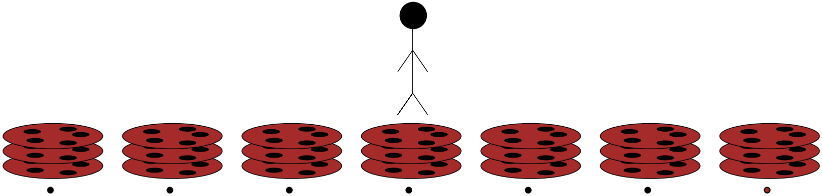


Conjecture

The ERRW is recurrent in $\mathbb{Z}^2$ for any reinforcement $c > 0$.

Excited (Cookie) Random Walks

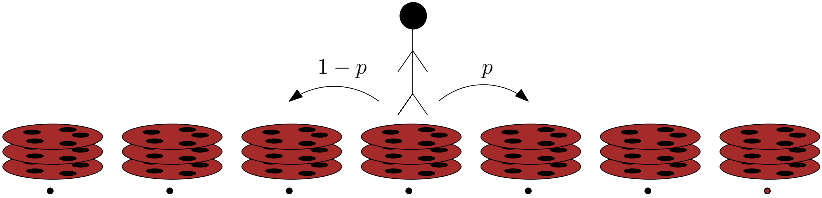
(M, p) **Cookie Random Walk**
Initially M cookies at each site.



Excited (Cookie) Random Walks

(M, p) **Cookie Random Walk**
Initially M cookies at each site.

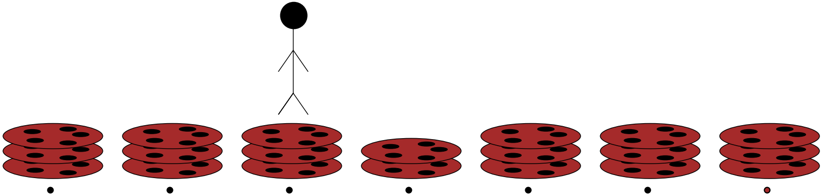
- **Cookie available:** Eat cookie. Move right with probability $p \in (0, 1)$



Excited (Cookie) Random Walks

(M, p) **Cookie Random Walk**
Initially M cookies at each site.

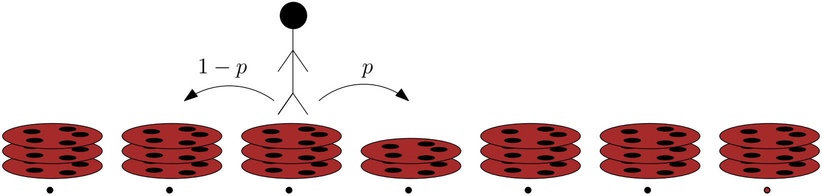
- **Cookie available:** Eat cookie. Move right with probability $p \in (0, 1)$



Excited (Cookie) Random Walks

(M, p) **Cookie Random Walk**
Initially M cookies at each site.

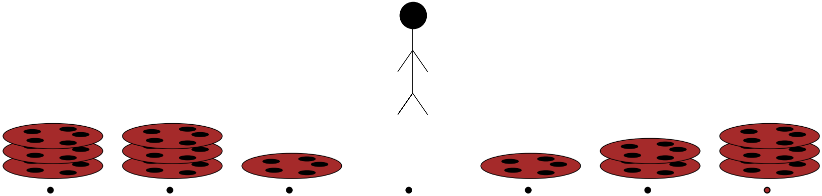
- **Cookie available:** Eat cookie. Move right with probability $p \in (0, 1)$



Excited (Cookie) Random Walks

(M, p) **Cookie Random Walk**
Initially M cookies at each site.

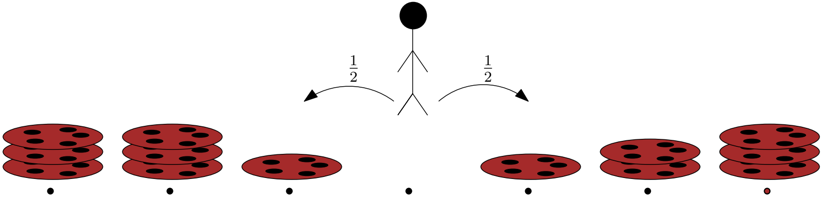
- **Cookie available:** Eat cookie. Move right with probability $p \in (0, 1)$
- **No cookies:** Move right/left with probability $\frac{1}{2}$.



Excited (Cookie) Random Walks

(M, p) **Cookie Random Walk**
Initially M cookies at each site.

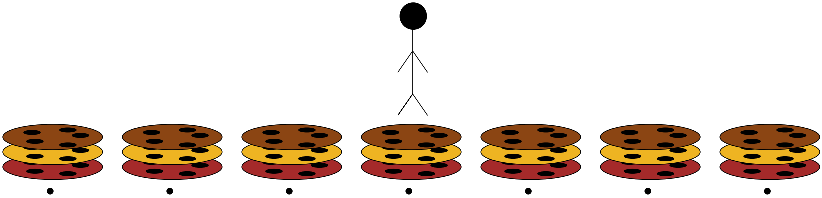
- **Cookie available:** Eat cookie. Move right with probability $p \in (0, 1)$
- **No cookies:** Move right/left with probability $\frac{1}{2}$.



Excited (Cookie) Random Walks

Unequal cookies

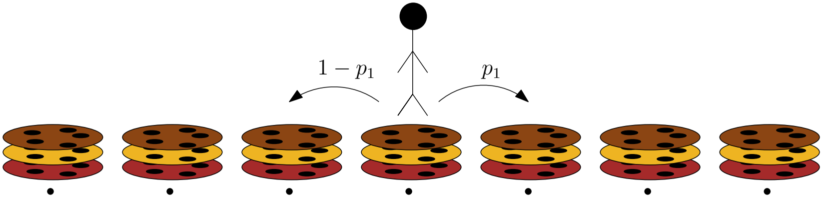
- M cookies at each site.
- Cookie *strengths* $p_1, p_2, \ldots, p_M \in (0,1)$.



Excited (Cookie) Random Walks

Unequal cookies

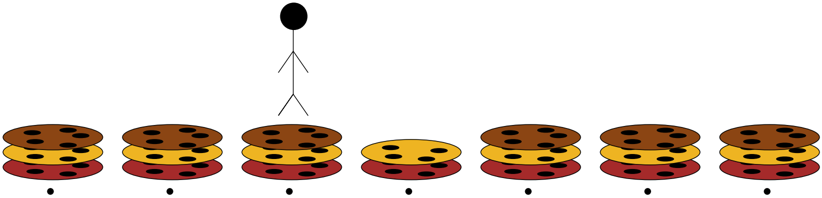
- M cookies at each site.
- Cookie *strengths* $p_1, p_2, \ldots, p_M \in (0,1)$.



Excited (Cookie) Random Walks

Unequal cookies

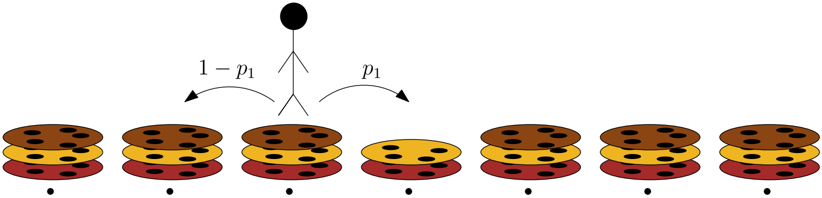
- M cookies at each site.
- Cookie *strengths* $p_1, p_2, \ldots, p_M \in (0,1)$.



Excited (Cookie) Random Walks

Unequal cookies

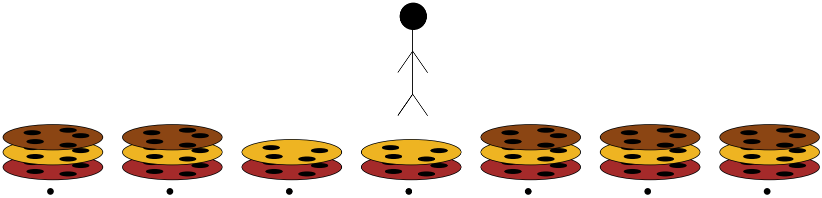
- M cookies at each site.
- Cookie *strengths* $p_1, p_2, \ldots, p_M \in (0,1)$.



Excited (Cookie) Random Walks

Unequal cookies

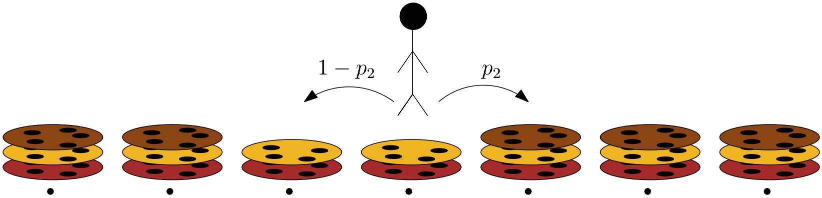
- M cookies at each site.
- Cookie *strengths* $p_1, p_2, \ldots, p_M \in (0,1)$.



Excited (Cookie) Random Walks

Unequal cookies

- M cookies at each site.
- Cookie *strengths* $p_1, p_2, \ldots, p_M \in (0,1)$.



Cookie Random Walks - Recurrence/Transience

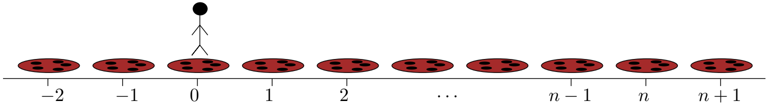
Question: Is the cookie random walk recurrent or transient?
Simple case: 1 cookie per site, strength $p > 1/2$.

When reaching n, what is the probability of reaching $n+1$ before returning to 0?

$$P(\text{never return to 0 after hitting } n) = \prod_{k=n}^{\infty} \left(1 - \frac{2(1-p)}{k}\right) = 0.$$

Cookie Random Walks - Recurrence/Transience

Question: Is the cookie random walk recurrent or transient?
Simple case: 1 cookie per site, strength $p > 1/2$.

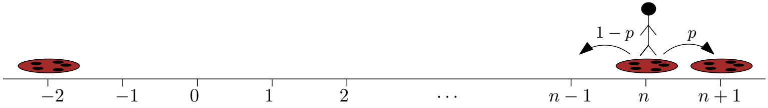


When reaching n, what is the probability of reaching $n+1$ before returning to 0?

$$P(\text{never return to 0 after hitting } n) = \prod_{k=n}^{\infty} \left(1 - \frac{2(1-p)}{k} \right) = 0.$$

Cookie Random Walks - Recurrence/Transience

Question: Is the cookie random walk recurrent or transient?
Simple case: 1 cookie per site, strength $p > 1/2$.

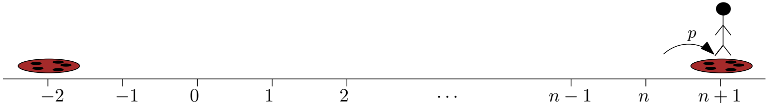


When reaching n, what is the probability of reaching $n+1$ before returning to 0?

$$P(\text{never return to 0 after hitting } n) = \prod_{k=n}^{\infty} \left(1 - \frac{2(1-p)}{k}\right) = 0.$$

Cookie Random Walks - Recurrence/Transience

Question: Is the cookie random walk recurrent or transient?
Simple case: 1 cookie per site, strength $p > 1/2$.



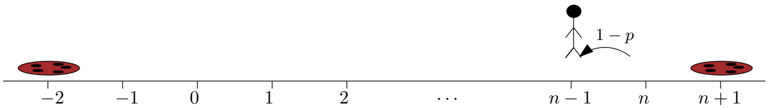
When reaching n, what is the probability of reaching $n+1$ before returning to 0?

p

$$P(\text{never return to 0 after hitting } n) = \prod_{k=n}^{\infty} \left(1 - \frac{2(1-p)}{k}\right) = 0.$$

Cookie Random Walks - Recurrence/Transience

Question: Is the cookie random walk recurrent or transient?
Simple case: 1 cookie per site, strength $p > 1/2$.



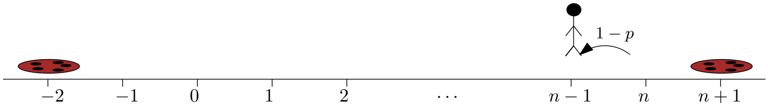
When reaching n, what is the probability of reaching $n+1$ before returning to 0?

$$p + (1-p)P_{\mathrm{SSRW}}(\text{reach } n+1 \text{ before } 0 \,|\, S_0 = n-1)$$

$$P(\text{never return to 0 after hitting } n) = \prod_{k=n}^{\infty}\left(1 - \frac{2(1-p)}{k}\right) = 0.$$

Cookie Random Walks - Recurrence/Transience

Question: Is the cookie random walk recurrent or transient?
Simple case: 1 cookie per site, strength $p > 1/2$.



When reaching n, what is the probability of reaching $n+1$ before returning to 0?

$$
\begin{aligned}
p + (1-p) P_{\mathrm{SSRW}}(\text{reach } n+1 \text{ before } 0 \,|\, S_0 = n-1) \\
= p + (1-p)\frac{n-1}{n+1} = 1 - \frac{2(1-p)}{n}
\end{aligned}
$$

$$
P(\text{never return to 0 after hitting } n) = \prod_{k=n}^{\infty} \left(1 - \frac{2(1-p)}{k}\right) = 0.
$$

Cookie Random Walks - Recurrence/Transience

Cookie RW with 1 cookie are always recurrent.
Question: When can a cookie RW be transient?
Key parameter: total drift per site

$$\delta = \sum_{i=1}^{M} (2p_i - 1).$$

Theorem (Zerner '05, Zerner & Kosygina '08)

The cookie random walk is

- *recurrent if $\delta \in [-1, 1]$.*
- *transient to the right if $\delta > 1$.*
- *transient to the left if $\delta < -1$.*

Cookie Random Walks - Limiting Speed

Question: Is there a limiting speed of the cookie RW?

$$\lim_{n\to\infty} \frac{X_n}{n} = v_0?$$

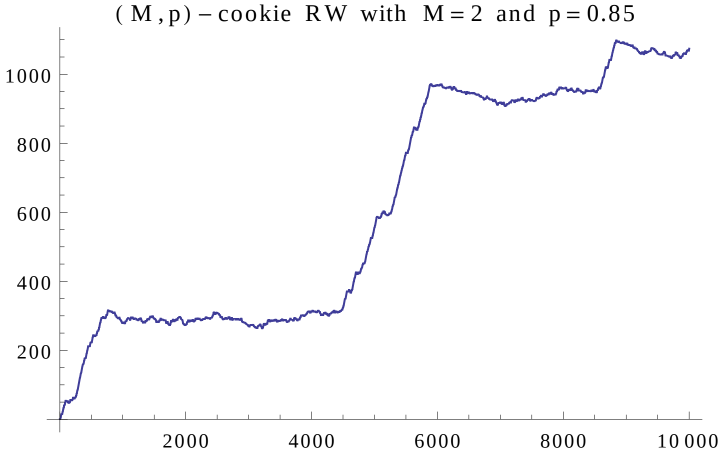
Theorem (Basdevant & Singh '07, Zerner & Kosygina '08)

The limiting speed $\lim_{n\to\infty} X_n/n = v_0$ *exists and is constant. Moreover*

- $v_0 > 0 \iff \delta > 2.$
- $v_0 = 0 \iff \delta \in [-2, 2].$
- $v_0 < 0 \iff \delta < -2.$

A transient cookie RW with "zero-speed"

An example with $\delta = 1.4$.



The limiting speed

Open Problem

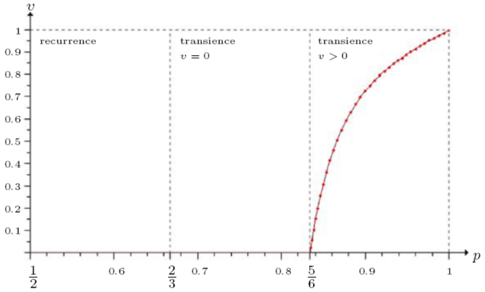
For a given cookie sequence $\vec{p} = (p_1, p_2, \dots, p_M)$ with $\delta = \delta(\vec{p}) > 2$, compute the limiting speed $v_0 = v_0(\vec{p})$.

The limiting speed

Open Problem

For a given cookie sequence $\vec{p} = (p_1, p_2, \ldots, p_M)$ *with* $\delta = \delta(\vec{p}) > 2$, *compute the limiting speed* $v_0 = v_0(\vec{p})$.

Open even for 3 cookies, all of strength $p > 5/6$.



Simulation from Basdevant and Singh

Monotonicity in the cookie sequence

Cookie sequences $\vec{p}$ and $\vec{q}$.

$$\vec{p} = (.85, .9, .95) \qquad \delta(\vec{p}) = 2.4$$
$$\vec{q} = (.8, .9, .95) \qquad \delta(\vec{q}) = 2.3$$

Monotonicity in the cookie sequence

Cookie sequences $\vec{p}$ and $\vec{q}$.

$$\vec{p} = (.85, .9, .95) \qquad \delta(\vec{p}) = 2.4$$
$$\vec{q} = (.8, .9, .95) \qquad \delta(\vec{q}) = 2.3$$

Question: Is it true that $v_0(\vec{p}) > v_0(\vec{q})$?

Monotonicity in simple random walks

Fix $q < p$

- X_n - simple random walk using q-coin
- Y_n - simple random walk using p-coin

Coupling: Construct the random paths $\{X_n\}_{n\geq 1}$ and $\{Y_n\}_{n\geq 1}$ so that

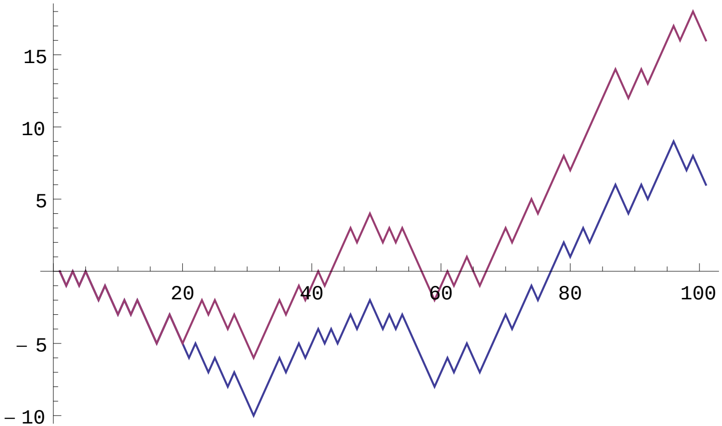
$$P(X_n \leq Y_n, \forall n \geq 0) = 1.$$

Let $U_1, U_2, \ldots$ be i.i.d. and Uniform(0,1).

$$X_n = X_{n-1} + \begin{cases} 1 & \text{if } U_n \leq q \\ -1 & \text{if } U_n > q. \end{cases} \quad \text{and} \quad Y_n = Y_{n-1} + \begin{cases} 1 & \text{if } U_n \leq p \\ -1 & \text{if } U_n > p. \end{cases}$$

Coupling of simple random walks

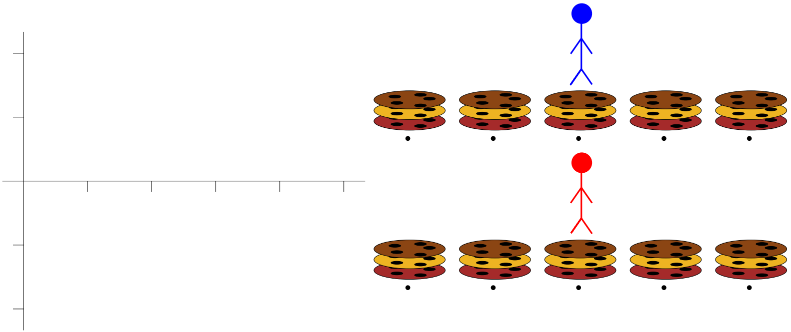
Coupled simple random walks with $q = .55 < .6 = p$.



Naive coupling of cookie random walks

Cookie sequences $\vec{p}$ and $\vec{q}$.

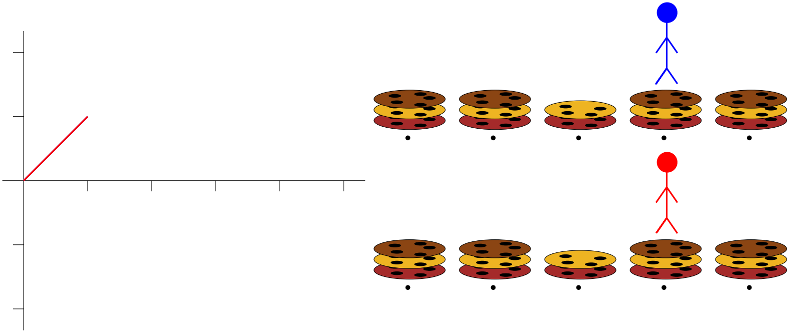
$$\color{blue}{\vec{p} = (.85, .9, .95)} \quad \text{and} \quad \color{red}{\vec{q} = (.8, .9, .95)}$$



Naive coupling of cookie random walks

Cookie sequences $\vec{p}$ and $\vec{q}$.

$$\color{blue}{\vec{p} = (.85, .9, .95)} \quad \text{and} \quad \color{red}{\vec{q} = (.8, .9, .95)}$$

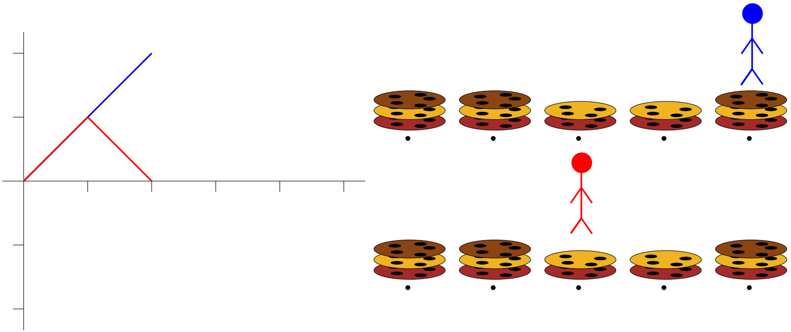


$U_1 = .6,$

Naive coupling of cookie random walks

Cookie sequences $\vec{p}$ and $\vec{q}$.

$$\color{blue}{\vec{p} = (.85, .9, .95)} \quad \text{and} \quad \color{red}{\vec{q} = (.8, .9, .95)}$$

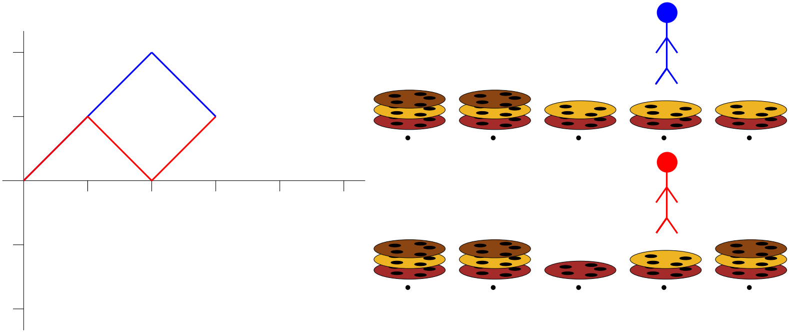


$U_1 = .6, \quad U_2 = .82,$

Naive coupling of cookie random walks

Cookie sequences $\vec{p}$ and $\vec{q}$.

$$\vec{p} = (.85, .9, .95) \quad \text{and} \quad \vec{q} = (.8, .9, .95)$$

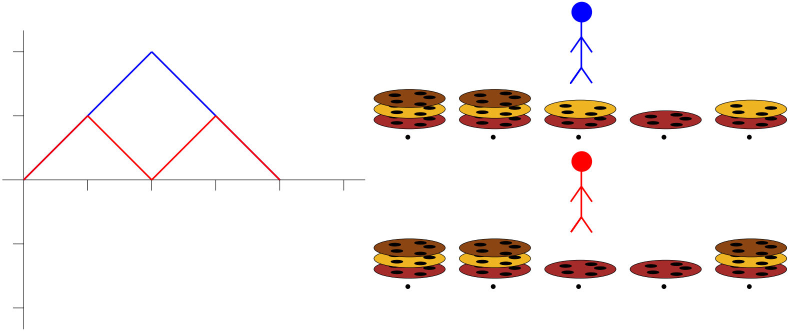


$U_1 = .6, \quad U_2 = .82, \quad U_3 = .87,$

Naive coupling of cookie random walks

Cookie sequences $\vec{p}$ and $\vec{q}$.

$$\vec{p} = (.85, .9, .95) \quad \text{and} \quad \vec{q} = (.8, .9, .95)$$

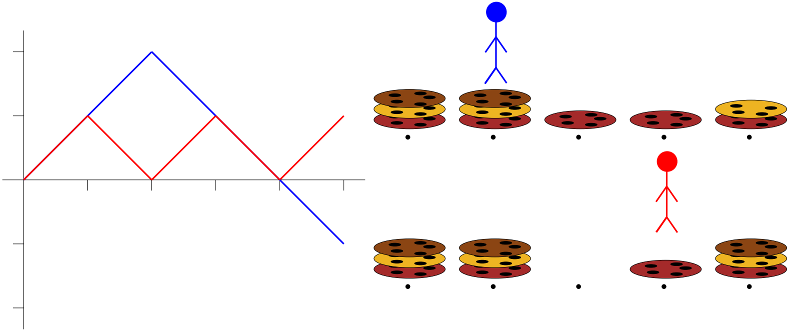


$$U_1 = .6, \quad U_2 = .82, \quad U_3 = .87, \quad U_4 = .3,$$

Naive coupling of cookie random walks

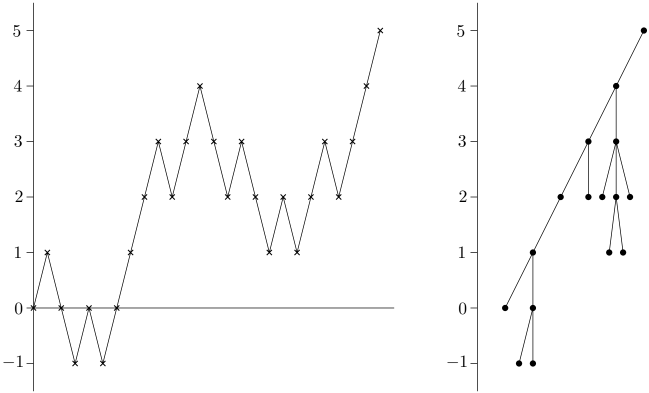
Cookie sequences $\vec{p}$ and $\vec{q}$.

$$\vec{p} = (.85, .9, .95) \quad \text{and} \quad \vec{q} = (.8, .9, .95)$$

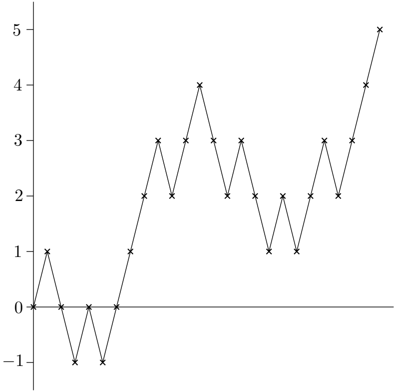


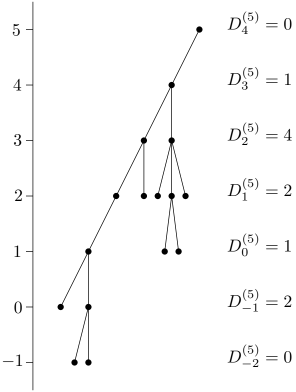
$$U_1 = .6, \quad U_2 = .82, \quad U_3 = .87, \quad U_4 = .3, \quad U_5 = .92$$

Associated branching process with migration

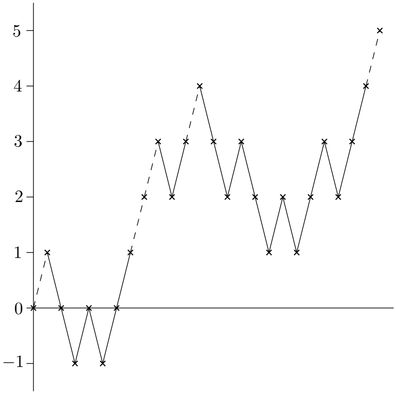


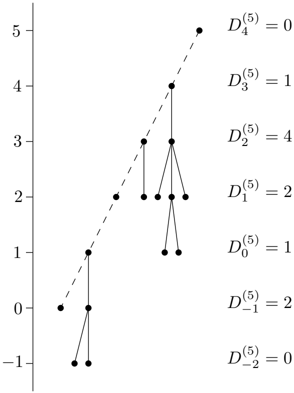
Associated branching process with migration





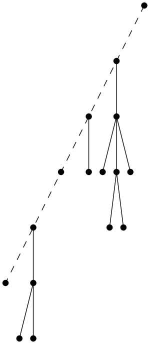
Associated branching process with migration





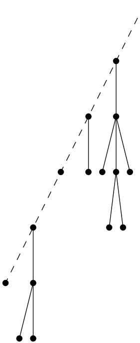
$$T_n = n + 2\sum_{i<n} D_i^{(n)}$$

Branching process offspring distribution



Simple random walk:
Offspring $\sim$ Geometric(p).

Branching process offspring distribution

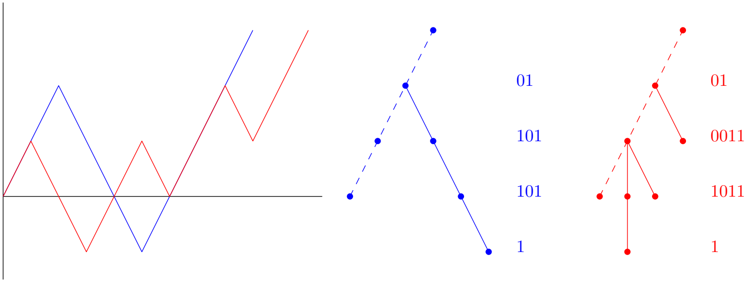


Simple random walk:
Offspring $\sim$ Geometric(p).

Cookie random walk:
Use independent
(Ber(p_1), Ber(p_2),. . . , Ber(p_M), Ber($1/2$), Ber($1/2$),. . .)
to generate offspring.

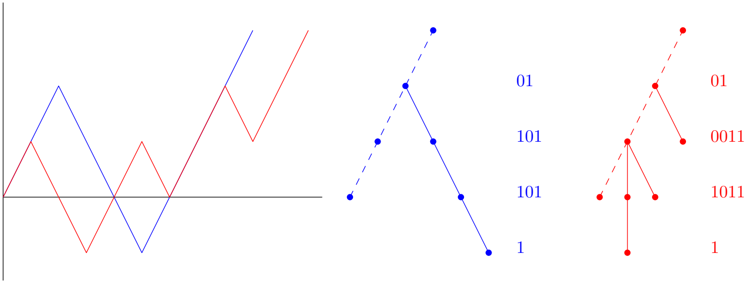
Coupling the branching processes

$\vec{p} = (.85, .9, .95)$ and $\vec{q} = (.8, .9, .95)$



Coupling the branching processes

$$\vec{p} = (.85, .9, .95) \quad \text{and} \quad \vec{q} = (.8, .9, .95)$$

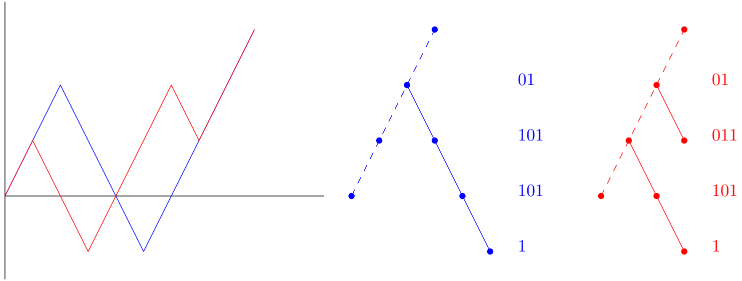


With this coupling, $T_n \leq T_n$.

Coupling the branching processes

Needed for coupling - partial sums of Bernoulli random variables.

$$\vec{p} = (.9, .8, .95) \quad \text{and} \quad \vec{q} = (.8, .9, .95)$$



Monotonicity of the speed for cookie RW

1. Relate hitting times to limiting speed.

$$\lim_{n\to\infty} \frac{T_n}{n} = \frac{1}{v_0}$$

Monotonicity of the speed for cookie RW

1. Relate hitting times to limiting speed.

$$\lim_{n\to\infty} \frac{T_n}{n} = \frac{1}{v_0}$$

2. Apply the coupling of branching processes to hitting times

$$\frac{1}{v_0} = \lim_{n\to\infty} \frac{T_n}{n} \leq \lim_{n\to\infty} \frac{T_n}{n} = \frac{1}{v_0}$$