

Research Statement

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My research area is probability theory. Most of my research has been in random walks in random environments (RWRE), but I am also interested in other non-classical random walks (such as excited random walks) and other stochastic processes (in particular interacting particle systems). Below, I will highlight some of my previous research results and also mention a few areas where I plan to do some research in the future.

1 Random walk in a random environment

A random walk in a random environment X_n is a Markov chain where the transition probabilities are also random. The random transition probabilities are considered the random *environment* in which the Markov chain evolves. This is most easily explained in the case of a simple RWRE on \mathbb{Z} in which case an environment $\omega = \{\omega_x\}_{x \in \mathbb{Z}}$ is a sequence of numbers in $[0, 1]$. For a fixed environment ω , X_n is a Markov chain that moves from x to $x + 1$ with probability ω_x and from x to $x - 1$ with probability $1 - \omega_x$.

Since there are two levels of randomness in the construction of a RWRE (the environment and then the walk), there are two different laws for the random walk that can be studied. The law of the random walk given the environment ω is called the *quenched* law and is denoted P_ω . The *averaged* (or *annealed*) distribution \mathbb{P} of the random walk is obtained by averaging the quenched distribution over all possible environments – that is $\mathbb{P}(\cdot) = \int P_\omega(\cdot) P(d\omega)$. The distribution P on environments is assumed to be stationary and ergodic under spatial shifts. A special case (which is assumed quite often) is when the environment is i.i.d. – that is, $\{\omega_x\}$ are independent and identically distributed random variables.

1.1 Background material

Random walks in a random environment have been shown to exhibit very interesting and surprising behavior. In Solomon’s seminal paper on one-dimensional RWRE [Sol75], he identified when a RWRE is recurrent/transient and he computed an explicit formula for the limiting velocity $V = \lim_{n \rightarrow \infty} X_n/n$. From these results it is easy to construct explicit examples of RWRE that are transient to $+\infty$ even though $\mathbb{E}X_1 < 0$ or that are transient with asymptotically zero velocity ($V = 0$). Soon after Solomon’s paper, Kesten, Kozlov, and Spitzer [KKS75] proved averaged limiting distributions for transient one-dimensional RWRE in i.i.d. environments. The centering, scaling, and limiting distribution are characterized by a parameter $\kappa = \kappa(P)$ of the distribution on environments by the following table.

κ	Centering	Scaling	Limiting Distribution Type
$0 < \kappa < 1$	0	n^κ	Stable(κ) ^{$-\kappa$}
$1 < \kappa < 2$	nV	$n^{1/\kappa}$	Stable(κ)
$\kappa > 2$	nV	\sqrt{n}	Gaussian

(1)

The borderline cases of $\kappa = 1, 2$ are also described in [KKS75], but the three regimes in (1) capture the main differences in behavior: a zero-speed regime, a positive-speed but non-Gaussian regime, and a Gaussian regime.

While multi-dimensional RWRE are not nearly as well understood as one-dimensional RWRE, there has been significant progress in recent years. In particular, there are a variety of conditions that are known to imply both a law of large numbers with a constant limiting velocity V and also a central limit theorem (see [Zei04] or [Szn04] for a survey of results).

1.2 One dimensional RWRE - Trapping effects

The dominant feature of one-dimensional RWRE is the existence of *traps*. A RWRE that is transient to $+\infty$ may still have sites that have drift to the left. Long segments of sites that have drift to the left serve to slow down (or trap) the random walk. Much of my work in one-dimensional RWRE has been done by analyzing how long it takes the random walk to escape these traps. The location of traps (also called valleys) in an environment ω are identified using what is called the potential of the environment which was first introduced by Sinai in his analysis of recurrent RWRE [Sin83].

In my Ph.D. thesis [Pet08], I studied the quenched distribution of a RWRE in the three regimes identified in (1). The simplest regime is the Gaussian regime ($\kappa > 2$) where the following quenched central limit theorem holds:

$$\lim_{n \rightarrow \infty} P_\omega \left(\frac{X_n - nV + Z_n(\omega)}{\sigma_1 \sqrt{n}} \leq x \right) = \Phi(x), \quad P - a.s. \quad (2)$$

The most interesting feature of the quenched central limit theorem is the existence of the environment-dependent correction $Z_n(\omega)$ to the deterministic centering nV . This result was also proved independently by Goldsheid, but in my thesis I was able to improve (2) to a quenched functional central limit theorem.

The main results of my Ph.D. thesis concerned the quenched distributions in the other two regimes of (1). Surprisingly, I was able to show that there does not exist a quenched limiting distribution when $\kappa < 2$. In fact, I showed in [Pet09] and [PZ09b] that for almost every environment ω there exist random sequences $t_k = t_k(\omega)$ and $t'_k = t'_k(\omega)$ such that X_{t_k} and $X_{t'_k}$ have different limiting distributions. The main idea of the proofs is that the quenched distribution is essentially determined by the amount of time it takes the random walk to spend in the few largest traps of the environment ω in $[0, n]$. When $\kappa < 2$, the relative sizes of the largest traps fluctuate wildly enough so that the quenched distribution fluctuates between mixtures of exponential distributions and never converges to any deterministic distribution.

I also used the method of trap analysis in a recent paper with Nina Gantert on bridges of one-dimensional RWRE [GP10] (i.e., RWRE conditioned to be back at the origin after $2n$ steps). For a classical simple random walk (i.e., $\omega_x \equiv p$) it is well known that when scaled by \sqrt{n} the distribution of bridges converges in distribution to the Brownian bridge. Thus, the maximal displacement from the origin of a bridge of a simple random walk is of the order \sqrt{n} . Interestingly, for RWRE that are transient to $+\infty$ and have mixed drifts (i.e., $P(\omega_0 > 1/2) > 0$ and $P(\omega_0 < 1/2) > 0$) the maximal displacement from the origin of a bridge of length $2n$ is roughly $n^{\kappa/(\kappa+1)}$ where κ is the parameter that also appeared as a scaling parameter for the averaged limiting distributions in (1).

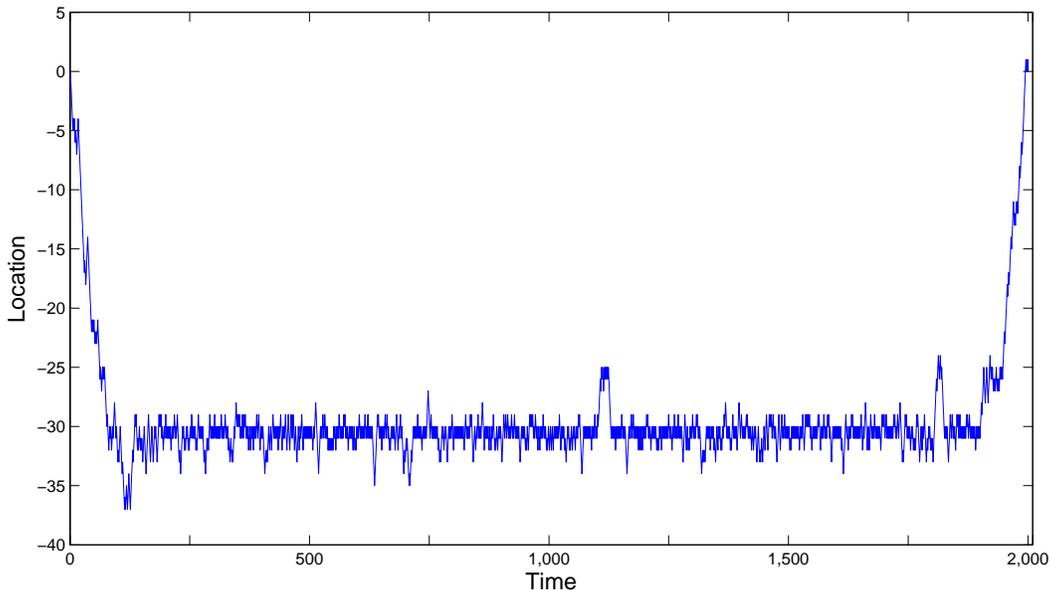


Figure 1: A simulation of a RWRE bridge of length 2000. The random walk spends most of its time in a small trap near $x = -30$.

It had been observed previously that the traps in the environment were the dominant feature for large and moderate deviations of slowdowns [PP99, FGP10]. Using similar techniques we were able to show that a RWRE bridge typically consists of the random walk going quickly to a large trap, staying in that trap for almost $2n$ steps, and then going quickly back to the origin (see Figure 1). The magnitude of the maximal displacement of bridges is a result of balancing the benefit of travelling farther from the origin to find a bigger trap with the cost of then having to backtrack a large distance back to the origin.

1.3 Large Deviations for RWRE

For one-dimensional RWRE, quite a lot is known about large deviations. There are both quenched and averaged large deviation principles as well as detailed qualitative descriptions of the quenched and averaged rate functions. One interesting fact is that while the rate functions for classical random walks are always strictly convex, the quenched and averaged rate functions for RWRE can be linear on segments near the origin. In [Var03], Varadhan proved both a quenched and averaged large deviation principle, but he was able to obtain very little information on the qualitative descriptions of the quenched and averaged rate functions $I_q(v)$ and $I_a(v)$ other than a description of where the rate functions are zero and the fact that the rate functions are both convex.

Recently, Ofer Zeitouni and I [PZ09a] and Atilla Yilmaz [Yil09] have independently obtained an improved qualitative descriptions of the averaged large deviation rate function in a neighborhood of the limiting velocity V . Our approach in [PZ09a] was to use the i.i.d. structure of what are called *regeneration times* of the random walk to obtain a new formulation for the averaged rate function $I_a(v)$ in an open set containing the limiting velocity V .

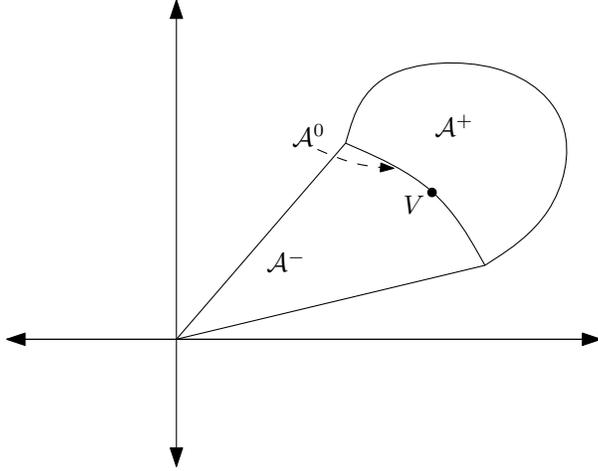


Figure 2: A visual depiction of the set $\mathcal{A} = \mathcal{A}^+ \cup \mathcal{A}^0 \cup \mathcal{A}^-$ where one can describe the qualitative behavior of the averaged rate function $I_a(v)$ in the nestling case. The rate function is strictly convex and analytic on \mathcal{A}^+ and the limiting velocity of the random walk $V \in \mathcal{A}^0 \subset \partial\mathcal{A}^+$. The rate function is 1-homogeneous on \mathcal{A}^- (that is $I_a(tv) = tI_a(v)$ for $t \in [0, 1]$ and $v \in \mathcal{A}^0$).

This new formulation leads to a qualitative description of the rate function. Most interesting is the fact that for certain environments (*nestling* environments that satisfy Sznitman's condition (T)) $I_a(v)$ is 1-homogeneous in a cone with vertex at the origin (see Figure 2).

2 Other Processes in Random Environments

2.1 Systems of RWRE

A recent area of interest for me has been studying other random processes in random environments. The first such process that I studied was a system of independent random walks in a common random environment. Starting with $\eta_0(x)$ particles at location $x \in \mathbb{Z}$ (η_0 may be deterministic or random) we let all the particles move independently as RWRE in a common environment ω and denote by $\eta_n(x)$ the number of particles at location x after n steps.

In [Pet10b], I obtained a couple of interesting results on the behavior of this system of independent RWRE. The first main result I obtained was a hydrodynamic limit for the system of particles. The hydrodynamic limit describes the configuration of particles at time n when space is also scaled by n . The second main result I obtained in [Pet10b] was a description of the limiting distribution of particles when there is no spatial scaling. For a natural class of initial configurations, the configuration η_n converges in distribution to a product Poisson distribution where the Poisson parameter at site x is an explicit function of x and the environment ω .

In particle systems, the fluctuations of the system from the deterministic configuration given by the hydrodynamic limit are often studied by analyzing what is called the current process of the system. The current is the net flow of particles observed by an independent

observer moving at the deterministic speed given by the hydrodynamic limit (or law of large numbers). Previous work of Kumar [Kum08] had shown that for a system of independent classical random walks, the current process, when scaled by $n^{1/4}$, converges to a centered Gaussian process with an explicit covariance matrix. In [PS10], Timo Seppäläinen and I analyzed the current process of a system of independent RWRE in the Gaussian regime (i.e., when the parameter $\kappa > 2$ in (1)), and showed that the fluctuations of the current process are of the order \sqrt{n} . The main contribution to the current is the environment-dependent centering term $Z_n(\omega)$ from the quenched central limit theorem (2), and if the current process is adjusted to account for this random centering term then the fluctuations are again of the order $n^{1/4}$.

2.2 Contact Process in a Random Environment

Another process that I have recently studied in a random environment is the contact process. The contact process is a simple model for the spread of an infection in a population. The standard model for the contact process on a graph is a continuous time Markov chain with the following dynamics: vertices that are infected become healthy at rate 1, and vertices that are healthy become infected at rate λ times the number of neighboring vertices that are infected. Typically one is interested in how the parameter λ changes the behavior of the process. Often there exists a constant λ_c called the critical value that marks the existence of a phase transition. If $\lambda < \lambda_c$ the contact process dies out quickly, but if $\lambda > \lambda_c$ the contact process survives for a long time with positive probability.

The standard graphs that the contact process is studied on are homogeneous deterministic graphs such as the d -dimensional integer lattice, the discrete torus, or d -regular trees. Recently there has been some work on the contact process on power law random graphs (random graphs where the degree distribution of a vertex has power law tails). Physicists first studied this model using non-rigorous mean-field methods and stated that for certain power law random graphs the critical parameter $\lambda_c > 0$ [PSV01b, PSV01a]. However, rigorous analysis by mathematicians showed that in fact the critical value λ_c is always $\lambda_c = 0$ [CD09, BBCS05].

Inspired by these results, I considered a slightly different model. Instead of using a random graph I studied the contact process on the complete graph, but I made the infection rates random. Let w_i be an i.i.d. sequence of weights on the vertices of the graph. Then the rate at which an infection travels between vertices i and j is $\lambda w_i w_j / n$ (the scaling by n is needed to make the maximal infection rate of order $\mathcal{O}(1)$). If the random vertex weights w_i have power law tails, a mean-field treatment of the model leads to the exact same equations as in the case of the power law random graphs. However, in [Pet10a] I was able to show that in this case the behavior of the contact process is consistent with a mean-field analysis of the model. In particular, I was able to compute explicit formulas for several quantities of interest (such as the critical value λ_c) that agreed with those arising from the corresponding mean-field analysis.

3 Future research

Some of my plans for future research include the following:

RWRE:

I would like to extend a few of my previous results in RWRE. Two such possible extensions are:

- Analyze bridges of RWRE that are recurrent.
- Provide a qualitative description of the multi-dimensional large deviation rate function $I_a(v)$ in a neighborhood of the origin.

In addition to these projects, there are many other areas open for research in RWRE - especially in higher dimensions. One such problem is what is known as the 0-1 law:

$$\text{Is } \mathbb{P}(\lim_{n \rightarrow \infty} X_n \cdot \ell = +\infty) \in \{0, 1\} \text{ for all } 0 \neq \ell \in \mathbb{R}^d ?$$

This seemingly simple question has turned out to be very difficult to answer, and surprisingly there are examples of non-i.i.d. environments where the 0-1 law doesn't hold [ZM01, BZZ06]. It is conjectured that the 0-1 law holds for RWRE in i.i.d. environments, but this has only been proved for 2-dimensional RWRE [ZM01]. One problem that I hope to work on is an extension of the 0-1 law for 2-dimensional RWRE to certain mixing environments.

Other non-classical random walks:

In addition to RWRE, I am also interested in other non-classical random walks such as excited random walks (also sometimes called cookie random walks). In the most basic model for excited random walks, the random walk moves to the right with probability $p > 1/2$ (and left with probability $1-p$) the first M times it visits each site and at subsequent visits moves to the right (and left) with probability $1/2$. Some of the techniques that have been useful in analyzing one-dimensional RWRE have also been used to analyze excited random walks [BS08, KZ08]. Currently I am trying to use some of these techniques to analyze the large deviations of excited random walks in one dimension.

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