HW 2 Due Feb 5

Laplace's Equation.

Do problems 2. and 3. at the end of Chapter 2 of the text [E] and the following:

3. Let $f \in C^2(\mathbb{R}^n)$. Assume that

$$\int_{\mathbb{R}^n} |f(y)|(1+|y|)^{2-n}\,dy < \infty \qquad \text{for } n > 3,$$
$$\int_{\mathbb{R}^2} |f(y)|\ln(1+|y|)\,dy < \infty \qquad \text{for } n = 2.$$

Prove that $u(x) = \Phi * f(x) \in C^2(\mathbb{R}^n)$ and $-\Delta u(x) = f(x)$.

You should use the following outline.

i) Show that $u(x)$ is well defined for each $x$.

ii) For any $1 < R < \infty$ let $\varphi(x) \in C^\infty(\mathbb{R}^n)$, be a cut-off function such that

$$\begin{aligned} \varphi &= 1 && \text{on } B(0,4R), \\ &= 0 && \text{on } \mathbb{R}^n \setminus B(0,5R). \end{aligned}$$

Write

$$u(x) = u_1(x) + u_2(x) = \Phi * (\varphi f)(x) + \Phi * ((1-\varphi)f)(x) \text{ for } x \in B(0,R).$$

Show that

$$u_2(x) = \int_{\{|y|\geq 3R\}} \Phi(x-y)(1-\varphi(y)f(y))\,dy. \text{ for } x \in B(0,R)$$

and

$$|\Phi(x+he_j-y)-\Phi(x-y)| \leq C(R)|y|^{1-n}|h| \text{ for } |y| \geq 3R, |x| \leq R, |h| \leq 1.$$

iii) Use the Dominated Convergence Theorem ([E]; Appendix E, Thm 5) (twice) to show

- $\partial_{x_j} u_2(x)$ exists on $B(0,R)$,
- $\partial_{x_j} u_2 \in C(B(0,R).)$

iv) Repeating the last step gives $u_2 \in C^\infty(B(0,R))$ and for any $\alpha$

$$D^\alpha u_2(x) = \int_{\{|y|\geq 3R\}} D^\alpha \Phi(x-y)(1-\varphi(y)f(y)\,dy. \text{ for } x \in B(0,R).$$

4. Let $u(x)$ be a $C^2$ solution to

$$-\Delta u(x) = |x|^2 \text{ on } \mathbb{R}^n.$$

- Show that $\Phi * |y|^2$ does not make sense.
- Find a solution nevertheless. Look for a polynomial.
- Set $m(r) = \fint_{\partial B(0,r)} u(y)\,dS(y)$

  Show that

$$m(r) = u(0) + \frac{r^4}{4(n+2)}.$$