The Ordinal Numbers and Transfinite Induction

Edward Price III

Purdue University

Student Colloquium
September 14, 2015

Table of Contents

Table of Contents

Cardinal Numbers

A cardinal number represents the *size* of a set.

The cardinality of a set S is denoted by $|S|$.

Examples

$|\varnothing| = 0$
$|\{0, 1, 2\}| = 3$
$|\mathbb{N}| = |\mathbb{Q}| = \aleph_0$, the first transfinite cardinal
$|\mathbb{R}| = \mathfrak{c}$, the cardinality of the continuum, strictly larger than $\aleph_0$

Intuitively, one would think to define the cardinal numbers as equivalence classes of sets, where $S \sim T$ if and only if there exists a bijection $f : S \to T$. This is how Bertrand Russell defined a "number."

Well-Ordered Sets

Definition

A set S is said to be **well-ordered** if S is totally ordered and every nonempty subset of S has a least member.

The canonical example of a well-ordered set is $\mathbb{N}$.

If we assume the Axiom of Choice (which we will in this talk), we can prove the Well-Ordering Theorem.

Well-Ordering Theorem

Every set can be well-ordered.

Well-Ordered Sets

Definition

Two well-ordered sets $(S, \leq)$ and $(T, \leq')$ are said to be **order isomorphic** if there exists a bijection $f : S \to T$ such that $s \leq s'$ if and only if $f(s) \leq' f(s')$.
In this case, f is said to be an **order isomorphism**.

Examples

- If $S = \{1, 2\}$ and $T = \{3, 15\}$ under the standard ordering, then $f : S \to T$ given by $f(1) = 3$ and $f(2) = 15$ is an order isomorphism.
- $g : \mathbb{N} \to \mathbb{N}_+$ by $g(n) = n + 1$ is an order isomorphism under the standard ordering.
- Let $\mathbb{N} \cup \{\omega\}$ be given the ordering that $n < \omega$ for all natural numbers n. Then there is no order isomorphism between $\mathbb{N}$ and $\mathbb{N} \cup \{\omega\}$.

Ordinal Numbers

An ordinal number can be thought of as the *position* of an element in a well-ordered set.

Example

Let $\mathbb{N} \cup \{\omega\}$ have the same ordering as before.
The 1st element is 0. The 14th element is 13. The *n*th element is $n - 1$.
The $(\omega + 1)$st element is ω.

An ordinal number can also be thought of as the *order type* of a well-ordered set (the position of the "last" element, if one exists).

Examples

The order type of $\varnothing$ is 0.
The order type of $\{0, 1, 2\}$ is 3.
The order type of $\mathbb{N}$ is ω.

Problems with Equivalence Classes

Like before, we could think of ordinal numbers as equivalence classes of well-ordered sets, where $(S, \leq) \sim (T, \leq')$ if and only if $(S, \leq)$ is order isomorphic to $(T, \leq')$.

This view of the ordinal numbers and the corresponding view of the cardinal numbers does *not* work in ZFC.

It turns out that bijection classes of sets are too big to be sets. Similarly, order isomorphism classes of well-ordered sets are too big to be sets.

This is related to Russell's Paradox. A collection $\mathcal{C}$ is too big to be a set if

$$\mathcal{C} \text{ a set} \Rightarrow \text{the set of all sets exists}$$

There are a few ways to prove that a collection $\mathcal{C}$ is too big to be a set.

$$\mathcal{C} \text{ a set} \Rightarrow \mathcal{C} \in \mathcal{C}$$

$$S \hookrightarrow \mathcal{C} \text{ for all sets } S$$

$$\mathcal{D} \text{ is too big to be a set and } \mathcal{D} \hookrightarrow \mathcal{C}$$

Table of Contents

The von Neumann Construction

The von Neumann construction of the natural numbers is as follows:
$0 := \varnothing$
$n + 1 := n \cup \{n\} = \{0, 1, 2, ..., n\}$

We can extend this construction to make a definition for every ordinal number.
An ordinal number α is defined to be the set of all ordinal numbers less than α.

Examples

$0 := \varnothing$
$5 := \{0, 1, 2, 3, 4\}$
$\omega := \mathbb{N} = \{0, 1, 2, ...\}$
$\omega + 1 := \mathbb{N} \cup \{\omega\} = \{0, 1, 2, ...; \omega\}$

Types of Ordinal Numbers

Definition

An ordinal number β is a **successor ordinal** if there exists an ordinal number α such that $\beta = \alpha + 1 = \{0, 1, ..., \alpha\}$.
An ordinal number λ is a **limit ordinal** if $\lambda > 0$ and for all ordinals α, $\lambda \neq \alpha + 1$.

Successor ordinals are named as such because they are the immediate successor of some other ordinal. Limit ordinals are named as such because they are the limit of the sequence of ordinals less then them without being a successor ordinal.

Examples

All natural numbers $n > 0$ are successor ordinals.
$\omega + n$ is a successor ordinal for all natural numbers $n > 0$.
ω is the smallest limit ordinal.

Order Type

Every well-ordered set is order isomorphic to exactly one ordinal number (and the isomorphism is unique!). As such, we make the following definition:

Definition

The **order type** of a well-ordered set $(S, \leq)$ is the unique ordinal number which is order isomorphic to $(S, \leq)$. Denote the order type of $(S, \leq)$ as $\mathbf{Ord}\,(S, \leq)$.

These facts allow us to see that this new definition of ordinal number fits in line with our intuitive notion of an ordinal number from before.

Cardinal Numbers

Even though every set can be well-ordered (Well-Ordering Theorem), for infinite sets, the order type can depend on the specific ordering chosen.

Examples

Ord $\mathbb{N} = \omega$ under the standard ordering.
If we choose an ordering $\leq'$ on $\mathbb{N}$ so that 0 is greater than every other natural number, but the other natural numbers have the standard ordering, then **Ord** $(\mathbb{N}, \leq') = \omega + 1 \neq \omega$.

With this in mind, we can define cardinality in the following way.

Definition

Let S be a set. The **cardinality** of S (denoted $|S|$) is the minimum order type of S under all possible well-orderings. i.e., $|S| = \min_{\leq} \textbf{Ord}\,(S, \leq)$.

We skip the proofs for time purposes, but one can prove that there exists a bijection between S and T iff $|S| = |T|$.

Table of Contents

Addition of Ordinal Numbers

Definition

Let α and β be ordinal numbers. Consider the set $\alpha \amalg \beta$. Define an well-order $\leq$ on $\alpha \amalg \beta$ by preserving the original orders on α and β themselves and having the additional criterion that all elements of β are greater than all elements of α.
Then $\alpha + \beta := \mathbf{Ord}\,(\alpha \amalg \beta, \leq)$.

Examples

$2 + 3 = \mathbf{Ord}\,(\{0, 1\} \amalg \{0, 1, 2\}) = \mathbf{Ord}\,\{0, 1, 0', 1', 2'\} = 5$
$\omega + 1 = \mathbf{Ord}\,(\mathbb{N} \amalg \{0\}) = \mathbf{Ord}\,\{0, 1, 2, ...; 0'\} = \omega + 1$
$1 + \omega = \mathbf{Ord}\,(\{0\} \amalg \mathbb{N}) = \mathbf{Ord}\,\{0, 0', 1', 2', ...\} = \omega$

The example of $\omega + 1$ justifies our choice of notation for successors.
The example of $1 + \omega$ shows that addition of ordinal numbers is not commutative!

Multiplication of Ordinal Numbers

Definition

Let α and β be ordinal numbers. Consider the set $\alpha \times \beta$. Define an well-order $<$ on $\alpha \times \beta$ by $(\alpha', \beta') < (\alpha'', \beta'')$ if and only if $\beta' < \beta''$ or $\beta' = \beta''$ and $\alpha' < \alpha''$. For convenience, we will denote (α', β') by $\alpha'_{\beta'}$
Then $\alpha\beta := \mathbf{Ord}\,(\alpha \times \beta, <)$.

Examples

$2 \cdot 3 = \mathbf{Ord}\,(\{0,1\} \times \{0,1,2\}) = \mathbf{Ord}\,\{0_0, 1_0, 0_1, 1_1, 0_2, 1_2\} = 6$
$\omega 2 = \mathbf{Ord}\,(\mathbb{N} \times \{0,1\}) = \mathbf{Ord}\,\{0_0, 1_0, 2_0, ...; 0_1, 1_1, 2_1, ...\} = \omega + \omega$
$2\omega = \mathbf{Ord}\,(\{0,1\} \times \mathbb{N}) = \mathbf{Ord}\,\{0_0, 1_0, 0_1, 1_1, 0_2, 1_2 ...\} = \omega$

Again, we see that multiplication is not commutative!

Ordinal Numbers are Complicated

So far, we have seen some operations and shown that they are quite complicated. For time purposes, we will skip ordinal exponentiation.
In ordinal exponentiation, we get even weirder results. For example, there are types of ordinal numbers called "ε-numbers" which satisfy the equation

$$\varepsilon = \omega^{\varepsilon}$$

The first ε-number is $\varepsilon_0 = \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}$
In this, we are dealing with unfathomably large numbers, but we've barely scratched the surface.
The collection of ordinal numbers is too big to be a set. Indeed, if it were a set, it would itself be an ordinal number, so it would have to contain itself as an element.
Overall, the ordinal numbers are very complicated, but they can be quite helpful when it comes to transfinite recursion and induction.

Table of Contents

Recursion and Induction

Recursion and Induction are similar processes.

Recursion *defines* a sequence of objects.

Induction *proves* a property is true for a sequence of objects.

Normally, we work with recursion and induction for only the natural numbers, giving us sequences of order type ω.

When we have sequences of order type larger than ω, we use *transfinite* recursion and induction, since our processes use transfinite ordinal numbers.

Transfinite Recursion

To define things recursively, we use the following process:

Transfinite Recursion

1. Base Case: Define an object S_0.

2. Successor Step: For all ordinal numbers α, define $S_{\alpha+1}$ using the definition of S_α.

3. Limit Step: For all limit ordinals λ, define S_λ using the definition of S_α for all ordinals $\alpha < \lambda$.

An Example

Consider $\mathbb{R}$ as a $\mathbb{Q}$-vector space. By Zorn's Lemma, we know this vector space must have a basis. Here, we will use transfinite recursion and induction to prove that this vector space has a basis.

Base Case: Let $B_0 = \varnothing$.

Successor Case: For all ordinals α, choose $v_{\alpha+1}$ to be an element of $\mathbb{R} - \text{Span } B_\alpha$. Then, define $B_{\alpha+1} = B_\alpha \cup \{v_{\alpha+1}\}$.

Limit Case: For all limit ordinals λ, define $B_\lambda = \bigcup_{\alpha<\lambda} B_\alpha$.

Since $|\mathbb{R}| < |\wp(\mathbb{R})| = \gamma$ for some ordinal γ, it follows that this process will terminate at some ordinal β, where $\mathbb{R} - \text{Span } B_\beta = \varnothing$.

Claim: B_β is a basis for $\mathbb{R}$ as a $\mathbb{Q}$-vector space.

To prove things inductively, we use the following process:

Transfinite Induction

Let P be a property and $P(\alpha)$ denote that P is true for the ordinal α. The following is the method of transfinite induction to prove that P is true for all ordinal numbers:

1. Base Case: Prove $P(0)$.
2. Successor Step: Assuming $P(\alpha)$ for arbitrary α, prove $P(\alpha + 1)$.
3. Limit Step: For all limit ordinals λ, assuming $P(\alpha)$ for all ordinals $\alpha < \lambda$, prove $P(\lambda)$.

Proof that B_β is a basis.

To show that B_β is a basis, it suffices to show that B_β is linearly independent. Afterall, $\mathbb{R} - \text{Span } B_\beta = \varnothing$, so B_β spans $\mathbb{R}$. To do this, it suffices to show that B_α is linearly independent for all ordinals α.

<u>Base case</u>: $B_0 = \varnothing$ is vacuously linearly independent.

<u>Successor case</u>: Suppose that for some ordinal α, B_α is linearly independent, but $B_{\alpha+1}$ is not. Now, $B_{\alpha+1} = B_\alpha \cup \{v_{\alpha+1}\}$, so it follows that $0 = c_1 w_1 + ... + c_n w_n + c_{n+1} v_{\alpha+1}$. We can then write

$$v_{\alpha+1} = -c_{n+1}^{-1} c_1 w_1 - ... - c_{n+1}^{-1} c_n w_n$$

So $v_{\alpha+1} \in \text{Span } B_\alpha$, which contradicts the fact that $v_{\alpha+1} \in \mathbb{R} - \text{Span } B_\alpha$. Thus, when B_α is linearly independent, $B_{\alpha+1}$ is linearly independent.

Proof that B_β is a basis.

Limit case: Suppose that λ is a limit ordinal. Suppose that B_α is linearly independent for all $\alpha < \lambda$.

Notice that, by definition, $B_\gamma \subset B_\delta$ for all ordinals $\gamma < \delta$. Hence, $\{B_\alpha\}_{\alpha<\lambda}$ is actually a chain of sets.

Now, suppose that B_λ is linearly dependent. Then there exists a finite set $\{w_1, ..., w_n\} \subset B_\lambda$ which is linearly dependent. By definition, $B_\lambda = \bigcup_{\alpha<\lambda} B_\alpha$, so it follows that there exist ordinals $\alpha_1, ..., \alpha_n < \lambda$ with $w_i \in B_{\alpha_i}$.

Since $\{B_\alpha\}_{\alpha<\lambda}$ is a chain, without loss of generality, it follows that $B_{\alpha_1} \subset ... \subset B_{\alpha_n}$. In particular, $\{w_1, ..., w_n\} \subset B_{\alpha_n}$, which contradicts the assumption that B_{α_n} is linearly independent (since $\alpha_n < \lambda$).

Therefore, B_λ is linearly independent whenever B_α is linearly independent for all $\alpha < \lambda$. $\square$

Every well-ordered set has an order type

We will now give another example of transfinite induction to prove that every well-ordered set has an order type. To do this, we will prove that every well-ordered set is uniquely order isomorphic to some ordinal.

Let $(S, \leq)$ be a well-ordered set. We will begin by using transfinite recursion to construct a sequence of subsets of $(S, \leq)$.

Base case: Let $S_0 = \varnothing$.

Successor case: For all ordinals α, define x_α to be the least element of $S - S_\alpha$, provided that $S - S_\alpha \neq \varnothing$. Then $S_{\alpha+1} = S_\alpha \cup \{x_\alpha\}$.

Limit case: For all limit ordinals λ, define $S_\lambda = \cup_{\alpha<\lambda} S_\alpha$.

This process must terminate for some ordinal β (where $S - S_\beta = \varnothing$). If not, then the ordinal numbers can be embedded into S, which makes S too big to be a set, contradicting the fact that S is a set.

Every well-ordered set has an order type

We prove that S_α is uniquely order isomorphic to α for all ordinals $\alpha \leq \beta$.

Base case: $S_0 = \varnothing$ is vacuously order isomorphic to $0 = \varnothing$ by the empty function, which is unique.

Successor case: Suppose S_α is uniquely order isomorphic to α for some ordinal $\alpha < \beta$. i.e., $f_\alpha : S_\alpha \to \alpha$ is the unique order isomorphism.

Now, by definition, $S_{\alpha+1} = S_\alpha \cup \{x_\alpha\}$, where x_α is the smallest element of $S - S_\alpha$. It follows that x_α is larger than all elements of S_α. So we establish the function $f_{\alpha+1} : S_{\alpha+1} \to \alpha + 1$ by $f_{\alpha+1}(x) = f_\alpha(x)$ for $x \in S_\alpha$ and $f_{\alpha+1}(x_\alpha) = \alpha$.

Clearly, $f_{\alpha+1}$ is an order isomorphism, and it must be unique since the maximal elements must be sent to each other and since f_α is unique.

Every well-ordered set has an order type

Limit case: Let $\lambda \leq \beta$ be a limit ordinal, and suppose that $f_\alpha : S_\alpha \to \alpha$ is the unique order isomorphism for all $\alpha < \lambda$. Notice that since $S_\gamma \subset S_\delta$ whenever $\gamma < \delta$, it follows that these order isomorphisms must agree. i.e., for all $\gamma < \delta$, $f_{\gamma+1}(x_\gamma) = f_\delta(x_\gamma)$. Now, define $f_\lambda : S_\lambda \to \lambda$ by $f_\lambda(x_\alpha) = f_{\alpha+1}(x_\alpha)$.

Suppose $f_\lambda(x_\gamma) = f_\lambda(x_\delta)$ for $\gamma \leq \delta < \lambda$. By definition, $f_\lambda(x_\gamma) = f_{\gamma+1}(x_\gamma)$ and $f_\lambda(x_\delta) = f_{\delta+1}(x_\delta)$. This gives the equality

$$f_{\gamma+1}(x_\gamma) = f_{\delta+1}(x_\delta)$$

Since the functions agree on subsets, $f_{\gamma+1}(x_\gamma) = f_{\delta+1}(x_\gamma)$. Then $f_{\delta+1}(x_\gamma) = f_{\delta+1}(x_\delta)$, so $x_\gamma = x_\delta$, and f_λ is injective.

Every well-ordered set has an order type

Now, let $\alpha \in \lambda$. Since λ consists of all ordinals less than λ, $\alpha < \lambda$. Since λ is a limit ordinal, $\alpha + 1 < \lambda$. Thus, $f_\lambda(x_\alpha) = f_{\alpha+1}(x_\alpha) = \alpha$, so f_λ is surjective.

Let $x_\gamma < x_\delta \in S_\lambda$. Then $f_\lambda(x_\gamma) = f_{\delta+1}(x_\gamma) < f_{\delta+1}(x_\delta) = f_\lambda(x_\delta)$, so f preserves order.

Lastly, f_λ is unique by the same argument as above (since f_λ is an order isomorphism, its restriction to S_α must be an order isomorphism with α, which is unique).

Hence, by transfinite induction, there is a unique order isomorphism from S_β to β. Since there is clearly (by construction) a unique order isomorphism from $(S, \leq)$ to S_β, it follows that there is a unique isomorphism from $(S, \leq)$ to β, meaning that $(S, \leq)$ has order type β. $\square$