MA 16020
Lesson 15
Improper Integrals

Sometimes, it is useful to consider the area under a curve from some point onwards. (This is often useful in probability, diff eqs, etc.) We consider integrals of the form $\int_a^\infty f(x)\,dx$, which are improper integrals with an infinite bound.

By definition, $\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$, if the limit exists. If the limit does not exist, we say the integral $\int_a^\infty f(x)dx$ diverges.

Ex 1. Find the integral, if it converges. $\int_1^\infty \frac{1}{3x-2}\,dx$

$$= \lim_{b\to\infty} \int_1^b \frac{1}{3x-2}\,dx = \lim_{b\to\infty} \frac{1}{3}\ln|3x-2|\Big|_1^b = \lim_{b\to\infty}\left(\frac{1}{3}\ln(3b-2) - \frac{1}{3}\ln(1)\right)$$

$$= \lim_{b\to\infty} \frac{1}{3}\ln(3b-2) = \infty$$ so the integral $\boxed{\text{diverges}}$

Ex 2. Find the integral, if it converges $\int_0^\infty \frac{5x}{e^{2x}}\,dx$

$$= \lim_{b\to\infty} \int_0^b 5xe^{-2x}\,dx \qquad u = 5x \quad v = -\frac{1}{2}e^{-2x}$$
$$du = 5\,dx \quad dv = e^{-2x}\,dx$$

$$\int 5xe^{-2x}dx = -\frac{5}{2}xe^{-2x} - \int -\frac{5}{2}e^{-2x}dx = -\frac{5}{2}xe^{-2x} - \frac{5}{4}e^{-2x}$$

So $$\lim_{b\to\infty}\left[-\frac{5}{2}xe^{-2x} - \frac{5}{4}e^{-2x}\right]_0^b = \lim_{b\to\infty}\left[-\frac{5}{2}be^{-2b} - \frac{5}{4}e^{-2b} + 0 + \frac{5}{4}\right]$$

By L'Hôpital's Rule, $\lim_{b\to\infty} \frac{-5b}{2e^{2b}} = \lim_{b\to\infty} \frac{-5}{4e^{2b}} = 0$

Also $\lim_{b\to\infty} -\frac{5}{4}e^{-2b} = 0$

So we get $0 - 0 + 0 + \frac{5}{4} = \frac{5}{4}$

Hence, $\int_0^\infty \frac{5x}{e^{2x}}\,dx = \boxed{\frac{5}{4}}$

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Ex 3. Find the integral, if it converges $\int_1^\infty \frac{e^{-2\sqrt{x}}}{2\sqrt{x}}\,dx$

$= \lim_{b\to\infty} \int_1^b \frac{e^{-2\sqrt{x}}}{\sqrt{x}}\,dx$ $\quad u = -2\sqrt{x} = -2x^{1/2}$, $du = -x^{-1/2}\,dx$
So $-du = \frac{1}{\sqrt{x}}\,dx$

$$\int \frac{e^{-2\sqrt{x}}}{2\sqrt{x}}\,dx = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u = -\frac{1}{2}e^{-2\sqrt{x}}$$

Thus, $\lim_{b\to\infty}\left[-\frac{1}{2}e^{-2\sqrt{x}}\right]_1^b = \lim_{b\to\infty}\left[-\frac{1}{2}e^{-2\sqrt{b}} + \frac{1}{2}e^{-2}\right] = 0 + \frac{1}{2}e^{-2}$

So $\int_1^\infty \frac{e^{-2\sqrt{x}}}{2\sqrt{x}}\,dx = \boxed{\frac{1}{2e^2}}$

There are also <u>improper integrals with an infinite discontinuity</u>. Such integrals are of the form $\int_a^b f(x)\,dx$ where $f(x)$ has an infinite discontinuity on $a \le x \le b$.

If $f(x)$ has an infinite discontinuity at a,
$\int_a^b f(x)\,dx = \lim_{s\to a^+} \int_s^b f(x)\,dx$, if it exists

If $f(x)$ has an infinite discontinuity at b,
$\int_a^b f(x)\,dx = \lim_{s\to b^-} \int_a^s f(x)\,dx$, if it exists.

Ex 4. Find the integral, if it converges $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

$\frac{1}{\sqrt{x}}$ has an infinite discontinuity at $x=0$

$$\lim_{s\to 0^+}\int_s^1 x^{-1/2}\,dx = \lim_{s\to 0^+}\left[2x^{1/2}\right]_s^1 = \lim_{s\to 0^+}\left[2s^{1/2} - 2\right]$$
$$= 0 - 2 = 2$$

Thus, $\int_0^1 \frac{1}{\sqrt{x}}\,dx = \boxed{2}$

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Ex 5. Find the integral if it converges $\int_0^{\pi} 6\tan\left(\frac{\theta}{2}\right) d\theta$

$\tan\left(\frac{\theta}{2}\right)$ has an infinite discontinuity when $\cos\left(\frac{\theta}{2}\right) = 0$

i.e., when $\frac{\theta}{2} = \frac{\pi}{2} + n\pi \Rightarrow \theta = \pi + 2n\pi$

issue is at π

$$\lim_{s\to\pi^-} \int_0^s 6\tan\left(\frac{\theta}{2}\right) d\theta = \lim_{s\to\pi^-} \left[-12\ln\left|\cos\left(\frac{\theta}{2}\right)\right|\right]_0^s$$

$$= \lim_{s\to\pi^-} \left[-12\ln\left|\cos\left(\frac{s}{2}\right)\right| + 0\right] = \lim_{t\to 0^+} -12\ln t = -\infty$$

So $\int_0^{\pi} 6\tan\left(\frac{\theta}{2}\right) d\theta$ $\boxed{\text{diverges}}$

A fun example. Find $\int_{-1}^{1} \frac{1}{x}\, dx$, if it exists

$\frac{1}{x}$ has an infinite discontinuity at $x = 0$

so $\int_{-1}^{1} \frac{1}{x}\, dx = \int_{-1}^{0} \frac{1}{x}\, dx + \int_0^1 \frac{1}{x}\, dx$

$$= \lim_{s\to 0^-} \int_{-1}^{s} \frac{1}{x}\, dx + \lim_{t\to 0^+} \int_t^1 \frac{1}{x}\, dx$$

$$= \lim_{s\to 0^-} \ln|x|\Big|_{-1}^{s} + \lim_{t\to 0^+} \ln|x|\Big|_t^1$$

$$= \underbrace{\lim_{s\to 0^-}\left(\ln|s| - \ln|-1|\right)}_{\substack{-\infty - 0 \\ \text{diverges!}}} + \underbrace{\lim_{t\to 0^+}\left(\ln|1| - \ln|t|\right)}_{\substack{0 + \infty \\ \text{diverges!}}}$$

So $\int_{-1}^{1} \frac{1}{x}\, dx$ diverges.

If you're not careful, you would do

$\int_{-1}^{1} \frac{1}{x}\, dx = \ln|x|\Big|_{-1}^{1} = \ln(1) - \ln|-1| = 0 - 0 = 0$

which is wrong