MA 16020
Lesson 21
Multivariate Differentials

Recall, for $y = f(x)$, $\frac{dy}{dx} = f'(x)$, so $dy = f'(x)\,dx$

Therefore, one can approximate change $\Delta y \approx f'(x)\,\Delta x$

For $z = f(x,y)$, we get partial differentials

$\partial z \approx \frac{\partial z}{\partial x} dx$ and $\partial z \approx \frac{\partial z}{\partial y} dy$ with respect to x and y.

Thus, we get the total differential

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

Thus $\Delta z \approx \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y$

Δvariable = (variable final value) − (variable initial value)

= change in variable.

Ex 1. Estimate $\ln(2.2^2 + 3.1) - \ln(2^2 + 3)$ using differentials. (Round 2 decimals)

Let $f(x,y) = \ln(x^2+y)$, $x = 2$, $y = 3$, the above expression is Δz.

$\Delta x = 2.2 - 2 = 0.2$, $\Delta y = 3.1 - 3 = 0.1$

$\frac{\partial f}{\partial x} = \frac{1}{x^2+y} 2x = \frac{2x}{x^2+y}$, $\frac{\partial f}{\partial y} = \frac{1}{x^2+y} \cdot 1 = \frac{1}{x^2+y}$

$$\Delta z \approx \frac{2x}{x^2+y} \Delta x + \frac{1}{x^2+y} \Delta y$$

$$= \frac{2(2)}{(2)^2+3}(0.2) + \frac{1}{(2)^2+3}(0.1) \approx \boxed{0.13}$$

Ex 2. An ideal gas satisfies $PV = 0.5T$, where P is pressure, V is volume, T is temperature. A scientist measures volume as $2\,m^3$ with an error of $0.1\,m^3$ and temperature as 200 Kelvin with an error of 5 Kelvin. What is the maximum error in the estimated pressure?

$P = 0.5TV^{-1}$, $\frac{\partial P}{\partial T} = 0.5V^{-1}$, $\frac{\partial P}{\partial V} = -0.5TV^{-2}$

$T = 200$, $V = 2$, $\Delta T = \pm 5$, $\Delta V = \pm 0.1$

$$\Delta P \approx \frac{0.5}{V}\Delta T + \frac{-0.5T}{V^2}\Delta V = \frac{0.5}{2}(\pm 5) + \frac{-0.5(200)}{2^2}(\pm 0.1)$$

$\approx -1.25, -3.75, 1.25,$ or 3.75

maximum is $\boxed{3.75 \text{ kPa}}$

Ex 3. A soup can with height h cm and radius r cm has volume $V = \pi r^2 h$. Currently, it has $h = 8$ cm and $r = 3$ cm. If they want to decrease the height by 0.1 cm and keep the volume the same, use differentials to estimate how much they should change the radius.

$\frac{\partial V}{\partial r} = 2\pi rh$, $\frac{\partial V}{\partial h} = \pi r^2$, $r = 3$, $h = 8$, $\Delta r = ?$, $\Delta h = -0.1$ (decrease), $\Delta V = 0$

$\Delta V \approx 2\pi rh\,\Delta r + \pi r^2\,\Delta h$

$0 \approx 2\pi(3)(8)\Delta r + \pi(3)^2(-0.1)$

$0 \approx 48\pi\,\Delta r - 0.9\pi$

$48\pi\,\Delta r \approx 0.9\pi$

$\Delta r \approx \frac{0.9}{48} \approx 0.019$

$\boxed{\text{increase radius about } 0.019 \text{ cm}}$

(because Δr is positive)

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Ex 4. A company has $P(x,y) = 20x^{3/4}y^{1/4}$ thousand units produced where x is number of employees and y is expenditures in thousands of dollars. Suppose they wish to reduce the employees from 100 to 90 and increase expenditures from \$15,000 to \$18,000.

(a) Estimate change in productivity due to change in employees

Partial differential $\partial z \approx \frac{\partial z}{\partial x}\partial x$

$\frac{\partial P}{\partial x} = 15x^{-1/4}y^{1/4}$, $\Delta P \approx 15x^{-1/4}y^{1/4}\Delta x$

$\Delta P \approx 15(100)^{-1/4}(15)^{1/4}(-10) \approx -93.350$ thousand units

decrease of 93.350 thousand units

(b) Estimate change in productivity due to change in expenditures.

$\frac{\partial P}{\partial y} = 5x^{3/4}y^{-3/4}$, $\Delta P \approx 5x^{3/4}y^{-3/4}\Delta y$

$\Delta P \approx 5(100)^{3/4}(15)^{-3/4}(3) \approx 62.233$

increase of 62.233 thousand units

(c) Estimate total change in productivity

Total differential is sum of the partial differentials

$\Delta P \approx -93.350 + 62.233 \approx -31.117$

decrease of 31.12 thousand units.

Ex 5. A can of height h cm and radius r cm is in production. The materials for the can cost 0.002 cents per cm^2 and the liquid inside costs 0.001 cents per cm^3. Estimate the change in cost by increasing height by .2 cm, decreasing radius by .3 cm if $r = 4$ cm and $h = 9$ cm.

$SA = 2\pi r^2 + 2\pi rh$, $V = \pi r^2 h$

$C = 0.002(2\pi r^2 + 2\pi rh) + 0.001\pi r^2 h$

$\frac{\partial C}{\partial r} = 0.002(4\pi r + 2\pi h) + 0.002\pi rh$, $\frac{\partial C}{\partial h} = 0.002(2\pi r) + 0.001\pi r^2$

$\Delta C \approx (0.002(4\pi r + 2\pi h) + 0.002\pi rh)\Delta r + (0.002 \cdot 2\pi r + 0.001\pi r^2)\Delta h$

$\Delta C \approx (0.002(4\pi(4) + 2\pi(9)) + 0.002\pi(4)(9))(-.3) + (0.002 \cdot 2\pi(4) + 0.001\pi(4)^2)(.2)$

≈ -0.112, so approx $\boxed{\text{0.112 cent decrease}}$

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Relative percent error of f

is $\dfrac{\text{max error of } f}{f} \cdot 100$

Ex 6. If $S = \dfrac{A}{A-W}$, A is measured to be 2.8 with max error of 0.01, W is measured to be 2.2 with max error of 0.05, approximate the relative percentage error of S.

$$\frac{\partial S}{\partial A} = \frac{(A-W)(1) - (A)(1)}{(A-W)^2} = \frac{-W}{(A-W)^2}$$

$$\frac{\partial S}{\partial W} = \frac{\partial}{\partial W}\left(A(A-W)^{-1} \right) = -A(A-W)^{-2}(-1) = \frac{A}{(A-W)^2}$$

$$\Delta S \approx \frac{-W}{(A-W)^2} \Delta A + \frac{A}{(A-W)^2} \Delta W$$

$$\text{error of } S \approx \frac{-2.2}{(2.8-2.2)^2}(\pm 0.01) + \frac{2.8}{(2.8-2.2)^2}(\pm 0.05)$$

$$\approx 0.328,\ 0.45,\ -0.328,\ -0.45$$

max error 0.45

Calculate $S = \dfrac{A}{A-W} = \dfrac{2.8}{2.8-2.2} = \dfrac{14}{3}$

relative percent error $= \dfrac{0.45}{\left(\frac{14}{3}\right)} \cdot 100 \approx \boxed{9.64\ \%}$