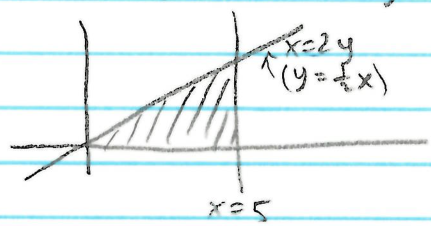
MA 16020
Lesson 29
Double Integrals (Part 3)

Remember some integrals require you to switch the order of integration.

Ex 1. $\int_0^{5/2}\int_{2y}^{5} 3e^{x^2}\,dx\,dy$

The inner integral is impossible.

x varies from $x=2y$ to $x=5$, y from $y=0$ to $y=5/2$



dydx: y varies from curve $y=0$ to $y=\frac{1}{2}x$
x varies from value $x=0$ to $x=5$

$$\int_0^5\int_0^{\frac{1}{2}x} 3e^{x^2}\,dy\,dx$$

$$= \int_0^5 3ye^{x^2}\Big|_0^{x/2}\,dx$$

$$= \int_0^5 \left(\tfrac{3}{2}xe^{x^2} - 0\right)dx$$

$u=x^2$, $du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

$u(0)=0^2=0$, $u(5)=5^2=25$

$$\int_0^{25} \tfrac{3}{4}e^u\,du$$

$$= \tfrac{3}{4}e^{25} - \tfrac{3}{4}e^0 = \boxed{\tfrac{3}{4}(e^{25}-1)}$$

MA 16020

Lesson 29

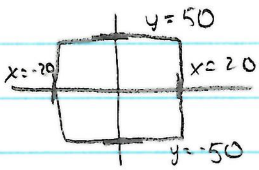
Average Value of $f(x,y)$

Sometimes we want the average value of $f(x,y)$ over a region R. This is given by the formula

$$\frac{1}{Area(R)} \iint_R f(x,y)\,dA$$

where Area(R) is the area of the region R which can sometimes be found using geometry of by using a single variable integral.

Ex 2. A building with a rectangular base has a curved roof whose height is $h(x,y) = 60 - 0.09x^2 + 0.021y^2$. The rectangular base extends from $-20 \le x \le 20$ feet and $-50 \le y \le 50$ feet. Find the average height of the building. (Round to 3 decimals)



$$\frac{1}{A} \int_{-50}^{50} \int_{-20}^{20} (60 - 0.09x^2 + 0.021y^2)\,dx\,dy$$

$$Area = (100)(40) = 4000$$

$$Ave = \frac{1}{4000} \int_{-50}^{50} \left(60x - 0.03x^3 + 0.021xy^2\right)\Big|_{-20}^{20} dy$$

$$= \frac{1}{4000} \int_{-50}^{50} \left(1200 - 240 + 0.42y^2 - (-1200 + 240 - 0.42y^2)\right) dy$$

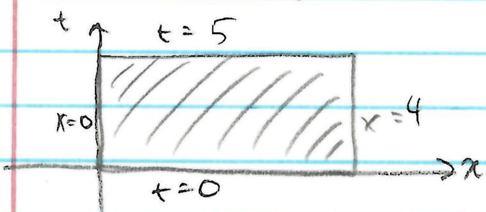
$$= \frac{1}{4000} \int_{-50}^{50} (1920 + 0.84y^2)\,dy$$

$$= \frac{1}{4000} \left[1920y + 0.28y^3\right]_{-50}^{50}$$

$$= \frac{1}{4000} \left[1920(50) + 0.28(50)^3 - 1920(-50) - 0.28(-50)^3\right]$$

$$= \frac{1}{4000} \left[262000\right] = \boxed{65.500 \text{ ft.}}$$

Ex 3. In a certain area, the population is $P(x,t) = \dfrac{8000e^{0.5t}}{1+x}$ where x is the number of miles from the center of the city, and t is the number of years after 2000. What is the average population over the first 5 years within a radius of 4 miles from the city center? (Round to whole number)



Area = (5)(4) = 20

$$\text{Ave} = \frac{1}{20}\int_0^4\int_0^5 \frac{8000e^{0.5t}}{1+x}\,dt\,dx$$

$$= \frac{1}{20}\int_0^4 \frac{16{,}000e^{0.5t}}{1+x}\Big|_0^4 dx$$

$$= \frac{1}{20}\int_0^4 \frac{16{,}000e^2 - 16{,}000}{1+x}\,dx$$

$$= \frac{16{,}000e^2 - 16{,}000}{20}\int_0^4 \frac{1}{1+x}\,dx$$

$$= \frac{16{,}000e^2 - 16{,}000}{20}\Big[\ln|1+x|\Big]_0^4$$

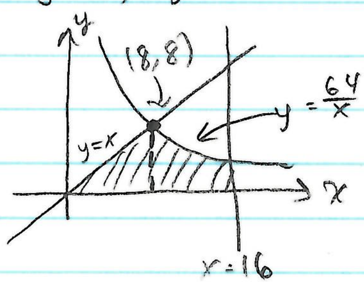
$$= \frac{16{,}000e^2 - 16{,}000}{20}\Big[\ln|5| - \ln|1|\Big]$$

$$\approx \boxed{8{,}226}$$

MA 16020

Lesson 29

Ex 4. Evaluate $\iint_R 10x^2\, dA$ over the region in the first quadrant bounded by $xy = 64$ and the lines $y = x$, $y = 0$ and $x = 16$.



Split into two regions

y from $y = 0$ to $y = x$

x from $x = 0$ to $x = 8$

and

y from $y = 0$ to $y = \frac{64}{x}$

x from $x = 8$ to $x = 16$

$$\int_0^8 \int_0^x 10x^2\, dy\, dx + \int_8^{16} \int_0^{64/x} 10x^2\, dy\, dx$$

$$= \int_0^8 10x^2 y \Big|_0^x dx + \int_8^{16} 10x^2 y \Big|_0^{64/x} dx$$

$$= \int_0^8 (10x^3 - 0)dx + \int_8^{16} (640x - 0)\, dx$$

$$= \frac{5}{2}x^4 \Big|_0^8 + 320x^2 \Big|_8^{16}$$

$$= \frac{5}{2}(8)^4 - \frac{5}{2}(0)^4 + 320(16)^2 - 320(8)^2$$

$$= \boxed{61600}$$