MA 16020

Lesson 30

Systems of Equations and Matrices

Systems of linear equations can look like

$$\begin{cases} x + 3y = 5 \\ 2x - y = 7 \end{cases} \quad \text{or} \quad \begin{cases} x - 5y + z = 4 \\ 3x \qquad + 2z = 5 \\ 7x - y - 4z = 8 \end{cases}$$

(or using more unknowns and more equations)

Linear systems can be solved using substitution of using elimination (or both). Notice how in elimination, you can multiply a single equation by a nonzero constant and you can add equations together. And you do this to eliminate a variable.

We can write systems of equations as augmented matrices, to help condense information.

Ex 1. Write the system as an augmented matrix.

$$\begin{cases} x - 5y + z = 4 \\ 3x \qquad + 2z = 5 \\ 7x - y - 4z = 8 \end{cases} \qquad \begin{array}{cc} & \begin{array}{cccc} x & y & z & = \end{array} \\ \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} & \left[\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 3 & 0 & 2 & 5 \\ 7 & -1 & -4 & 8 \end{array}\right] \end{array}$$

Based on the techniques of elimination, you can perform <u>Elementary Row Operations</u> on an augmented matrix, and it won't change the solutions of the system.

1. Swap two rows (e.g., $R_1 \leftrightarrow R_2$)
2. Multiply a row by a nonzero number (e.g., $\frac{1}{2}R_2 \to R_2$, $\frac{1}{2}R_2$ replacing R_2)
3. Adding a multiple of one row to another row (e.g., $2R_1 + R_2 \to R_2$, replacing R_2 with 2 time R_1 added to R_2)

Guassian Elimination

Given a system of equations,

1. Write the system as an augmented matrix
2. Perform elementary row operations to get the (1,1)-entry to be a 1 (the (i, j)-entry of a matrix is the entry in row i and column j)
3. Perform elementary row operations to get all entries below (1,1) to be 0.
4. Perform elementary row operations to get the (2,2)-entry to be a 1.
5. Perform elementary row operations to get all entries below (2,2) to be 0

   etc.

6. The process ends when your matrix is in Row Echelon Form $\left[\begin{array}{ccc|c} 1 & a & b & d \\ 0 & 1 & c & e \\ 0 & 0 & 1 & f \end{array}\right]$ or $\left[\begin{array}{cc|c} 1 & a & b \\ 0 & 1 & c \end{array}\right]$

   (1's along diagonal and 0's under the diagonal)

7. Now use substitution to solve your new system.

Ex 2. A goldsmith has two alloys of gold, the first having a purity of 70% and the second having a purity of 82%. If x grams of the first alloy are mixed with y grams of the second, obtaining 100 grams of alloy with 78% gold, find x to the nearest gram.

grams: $x+y=100$

percentage gold: $0.7x+0.82y=0.78\cdot 100$

$$0.7x+0.82y=78$$

$$\begin{cases} 0.7x + 0.82y = 78 \\ x + y = 100 \end{cases}$$

$$\left[\begin{array}{cc|c} 0.7 & 0.82 & 78 \\ 1 & 1 & 100 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 1 & 100 \\ 0.7 & 0.82 & 78 \end{array}\right]$$

$$\xrightarrow{-0.7R_1+R_2} \left[\begin{array}{cc|c} 1 & 1 & 100 \\ 0 & 0.12 & 8 \end{array}\right] \xrightarrow{\frac{1}{0.12}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 1 & 100 \\ 0 & 1 & \frac{200}{3} \end{array}\right] \text{ REF}$$

$$\begin{cases} x+y=100 \\ y=\frac{200}{3} \end{cases} \Rightarrow x+\frac{200}{3}=100 \Rightarrow x=100-\frac{200}{3} \approx \boxed{33 \text{ grams}}$$

Ex 3. An object is moving vertically where a is constant acceleration, v is velocity at $t=0$ and h is height at $t=0$ is represented by the equation $s(t)=\frac{1}{2}at^2+vt+h$. When $t=1$, $s=38$; when $t=2$, $s=50$; when $t=3$, $s=38$. Find the equation for $s(t)$.

$s(1)=\frac{1}{2}a+v+h=38$, $s(2)=\frac{1}{2}a(2)^2+v(2)+h=2a+2v+h=50$,
$s(3)=\frac{1}{2}a(3)^2+v(3)+h=\frac{9}{2}a+3v+h=38$

$$\begin{array}{c} \begin{array}{ccc} a & v & h \end{array} \\ \left[\begin{array}{ccc|c} \frac{1}{2} & 1 & 1 & 38 \\ 2 & 2 & 1 & 50 \\ \frac{9}{2} & 3 & 1 & 38 \end{array}\right] \end{array} \xrightarrow{2R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 76 \\ 2 & 2 & 1 & 50 \\ \frac{9}{2} & 3 & 1 & 38 \end{array}\right]$$

$$\xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 76 \\ 0 & -2 & -3 & -102 \\ \frac{9}{2} & 3 & 1 & 38 \end{array}\right] \xrightarrow{-\frac{9}{2}R_1+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 76 \\ 0 & -2 & -3 & -102 \\ 0 & -6 & -8 & -304 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 76 \\ 0 & 1 & \frac{3}{2} & 51 \\ 0 & -6 & -8 & -304 \end{array}\right] \xrightarrow{6R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 76 \\ 0 & 1 & \frac{3}{2} & 51 \\ 0 & 0 & 1 & 2 \end{array}\right] \text{ REF}$$

$$\begin{cases} a+2v+2h=76 \\ v+\frac{3}{2}h=51 \Rightarrow v+\frac{3}{2}(2)=51 \Rightarrow v=48 \\ h=2 \qquad a+2(48)+2(2)=76 \Rightarrow a=-24 \end{cases}$$

$s(t)=\frac{1}{2}(-24)t^2+48t+2$

$$\boxed{s(t)=-12t^2+48t+2}$$

A system is called inconsistent if it has no solutions (e.g., if you get $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 5 \end{array}\right]$)

It is called consistent independent if it has one solution (e.g., examples 2 + 3)

It is called consistent dependent if it has infinitely many solutions, then the solutions depend on the value of one of the variables.

e.g., if you end up with

$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 0 & 1 & 3 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right]$, then you have $\begin{cases} x + 2y + 2z = 4 \\ y + 3z = 8 \\ 0 = 0 \end{cases}$

Usually, we set $z = t$.

Then $y + 3t = 8 \Rightarrow y = 8 - 3t$

then $x + 2(8-3t) + 2t = 4$

$\Rightarrow x + 16 - 6t + 2t = 4$

$\Rightarrow x = 4t - 12$

So solutions are of the form

$\boxed{(4t-12,\ 8-3t,\ t)}$

Any value plugged in to t is a solution!