

A differential equation (diff eq) is an equation relating a function and its derivative(s).

For example, $y = y'$ is a diff eq. A solution to a diff eq is a function y which satisfies the equation. In the example $y = y'$, $y = Ce^x$ is a solution for any constant C .

A diff eq together with an initial condition is an initial value problem (IVP). The initial condition determines the value of C .

In the next 3 lessons, we study separable diff eqs. In such diff eqs, we can get all y 's on one side and all x 's or t 's on the other, then integrate.

Ex 1. Solve the diff eq $y' = e^{y-2x}$

$$\frac{dy}{dx} = e^y \cdot e^{-2x}$$

$$e^{-y} dy = e^{-2x} dx$$

$$\int e^{-y} dy = \int e^{-2x} dx$$

$$-e^{-y} = -\frac{1}{2}e^{-2x} + C$$

$$e^{-y} = \frac{1}{2}e^{-2x} + C$$

$$-y = \ln\left(\frac{1}{2}e^{-2x} + C\right)$$

$$\boxed{y = -\ln\left(\frac{1}{2}e^{-2x} + C\right)}$$

Ex 2. Solve the IVP $y' = 3 \ln t$, $y(1) = 7$

$$\frac{dy}{dt} = 3 \ln t, \text{ so } dy = 3 \ln t \, dt$$

$$y = \int dy = \int 3 \ln t \, dt$$

Try integration by parts

$$u = \ln t$$

$$dv = 3 \, dt$$

$$du = \frac{1}{t} \, dt$$

$$v = 3t$$

$$y = 3t \ln t - \int 3t \cdot \frac{1}{t} \, dt$$

$$y = 3t \ln t - 3t + C$$

$$7 = y(1) = 3(1) \ln(1) - 3(1) + C$$

$$7 = -3 + C, \quad C = 10$$

$$\boxed{y(t) = 3t \ln t - 3t + 10}$$

Growth and Decay

Growth and decay are accurately modeled through differential equations in which the rate of change of a function is proportional to the function.

If y' is (directly) proportional to y , then $y' = ky$ for some constant k .

If y' is inversely proportional to y , then $y' = \frac{k}{y}$ for some constant k .

If y' is jointly proportional to y and w , then $y' = kyw$, for some constant k .

(Skip?)

Ex 3. The rate of change of the number of turtles $N(t)$ in a population is directly proportional to $400 - N(t)$, where t is measured in years. When $t = 0$, the population is 300. When $t = 3$, the population is 380. Find the population when $t = 4$. (Round to whole number)

$$\frac{dN}{dt} = k(400 - N)$$

$$\int \frac{dN}{400 - N} = \int k dt \quad u = 400 - N, du = -dN$$

$$-\ln|400 - N| = kt + C$$

$$\ln|400 - N| = -kt + C$$

$$|400 - N| = e^{-kt+C} = e^{-kt} e^C = C e^{-kt}$$

$$400 - N = C e^{-kt}$$

$$N = C e^{-kt} + 400$$

$$300 = N(0) = C + 400, \text{ so } C = -100$$

$$N = -100 e^{-kt} + 400$$

$$380 = N(3) = -100 e^{-3k} + 400$$

$$-20 = -100 e^{-3k}$$

$$\frac{1}{5} = e^{-3k} \Rightarrow -3k = \ln\left(\frac{1}{5}\right) \Rightarrow k = -\frac{1}{3} \ln\left(\frac{1}{5}\right)$$

$$N(t) = -100 e^{\frac{1}{3} \ln\left(\frac{1}{5}\right)t} + 400$$

$$N(4) = -100 e^{\frac{1}{3} \ln\left(\frac{1}{5}\right) \cdot 4} + 400$$

$$\approx \boxed{388 \text{ turtles}}$$

Ex 4. A radioactive element decays proportionally to its amount. Its half-life is 4 years. If it starts with 30 mg, how much is left in 7 years? (Round to 2 decimal places)

$$\frac{dA}{dt} = kA$$

$$\int \frac{dA}{A} = \int k dt$$

$$\ln|A| = kt + C$$

$$A = Ce^{kt}$$

$$30 = A(0) = Ce^0 = C$$

$$A(t) = 30e^{kt}$$

$$15 = 30e^{4k}$$

$$\frac{1}{2} = e^{4k}$$

$$4k = \ln\left(\frac{1}{2}\right) \Rightarrow k = \frac{1}{4}\ln\left(\frac{1}{2}\right)$$

$$A(t) = 30e^{\frac{1}{4}\ln\left(\frac{1}{2}\right)t}$$

$$A(7) = 30e^{\frac{1}{4}\ln\left(\frac{1}{2}\right) \cdot 7} \approx \boxed{8.92 \text{ mg}}$$

Newton's Law of Cooling states that the temperature of an object changes at a rate proportional to the difference between the object's temperature and the temperature of the surrounding area, i.e.,

$$\frac{dT}{dt} = k(T_s - T)$$

where T_s is the surrounding temperature and $T(t)$ is temperature as a function of time.

Ex 5. After 6 minutes, the temperature of a cup of hot cocoa has decreased to 60°C from 100°C . The room has a temperature of 22°C . How much longer will it take to cool down to 40°C ?

$$\frac{dT}{dt} = k(22 - T) \quad (\text{Round to hundredth})$$

$$\frac{dT}{22 - T} = k dt$$

$$-\ln|22 - T| = kt + C$$

$$22 - T = Ce^{-kt}$$

$$T = 22 + Ce^{-kt}$$

$$100 = T(0) = 22 + C \Rightarrow C = 78$$

$$T(t) = 22 + 78e^{-kt}$$

$$60 = T(6) = 22 + 78e^{-6k}$$

$$\frac{38}{78} = e^{-6k} \Rightarrow k = -\frac{1}{6} \ln\left(\frac{19}{39}\right)$$

$$T(t) = 22 + 78e^{\frac{1}{6} \ln\left(\frac{19}{39}\right)t}$$

$$40 = 22 + 78e^{\frac{1}{6} \ln\left(\frac{19}{39}\right)t}$$

$$\frac{3}{13} = e^{\frac{1}{6} \ln\left(\frac{19}{39}\right)t}$$

$$\ln\left(\frac{3}{13}\right) = \frac{1}{6} \ln\left(\frac{19}{39}\right)t$$

$$t = \frac{6 \ln\left(\frac{3}{13}\right)}{\ln\left(\frac{19}{39}\right)} \approx 12.23 \text{ minutes}$$

Since how much longer

$$12.23 - 6 = \boxed{6.23 \text{ minutes}}$$