

MA 16020
Lesson 8
Separable Equations (Part 2)

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Ex 1. A 600 gallon tank initially contains 400 gallons of pure distilled water. Brine containing 3 pounds of salt per gallon flows into the tank at a rate of 4 gallons per minute, and the well-stirred mixture flows out of the tank at the same rate. Find the amount of salt in the tank after 20 minutes. (Round 2 decimal)

Let $S(t)$ be lbs of salt in tank after t minutes

$$\begin{aligned}\frac{dS}{dt} &= \text{rate of change of lbs of salt in the tank after } t \text{ minutes} \\ &= \left(\begin{array}{l} \text{rate of change of} \\ \text{lbs of salt into tank} \end{array} \right) - \left(\begin{array}{l} \text{rate of change of lbs of} \\ \text{salt out of tank} \end{array} \right) \\ &= \text{units (lbs/min)}\end{aligned}$$

$$\begin{aligned}&= \text{rate in (concentration of salt)} \cdot (\text{rate of flow}) \\ &= \left(\frac{3 \text{ lbs}}{\text{gal}} \cdot \frac{4 \text{ gal}}{\text{min}} \right) - \left(\frac{S \text{ lbs}}{400 \text{ gal}} \cdot \frac{4 \text{ gal}}{\text{min}} \right) \\ &= 12 - \frac{S}{100}\end{aligned}$$

$$\frac{1}{12 - \frac{1}{100}S} dS = dt$$

$$u = 12 - \frac{1}{100}S, \quad du = -\frac{1}{100}dS \Rightarrow -100 du = dS$$

$$-100 \int \frac{1}{u} du = \int dt$$

$$-100 \ln |12 - \frac{1}{100}S| = t + C$$

$$\ln |12 - \frac{1}{100}S| = -\frac{t}{100} + C$$

$$12 - \frac{1}{100}S = Ce^{-t/100}$$

$$-\frac{1}{100}S = Ce^{-t/100} - 12$$

$$S = Ce^{-t/100} + 1200$$

$$\text{when } t=0, S=0 \text{ (pure water), } C = -1200$$

$$S(20) = -1200e^{-20/100} + 1200 \approx \boxed{217.52 \text{ lbs}}$$

Ex 2. The rate of change in the number of miles of road cleared per hour by a snowplow is inversely proportional to the depth of the snow. Given that 30 miles per hour are cleared when the depth is 2 inches and 16 miles per hour are cleared when the depth is 8 inches, how many miles of road will be cleared per hour when the depth of snow is 12 inches? (Round 2 decimals)

M = miles cleared per hour
 s = depth of snow in inches

$$\frac{dM}{ds} = \frac{K}{s} \Rightarrow dM = \frac{K}{s} ds$$

$$\Rightarrow M = K \ln|s| + C$$

$$\begin{cases} 30 = K \ln(2) + C \\ 16 = K \ln(8) + C \end{cases} \Rightarrow \begin{aligned} 30 &= K \ln(2) + C \\ -16 &= -K \ln(8) - C \\ \hline 14 &= K \ln(2) - K \ln(8) \\ 14 &= K (\ln(2) - \ln(8)) \\ 14 &= K \ln\left(\frac{2}{8}\right) = K \ln\left(\frac{1}{4}\right) \end{aligned}$$

so $K = \frac{14}{\ln(\frac{1}{4})}$

$$16 = \frac{14}{\ln(\frac{1}{4})} \ln(8) + C, \text{ so } C = 16 - \frac{14 \ln(8)}{\ln(\frac{1}{4})}$$

$$M = \frac{14}{\ln(\frac{1}{4})} \ln(s) + 16 - \frac{14 \ln(8)}{\ln(\frac{1}{4})}$$

$$M(12) = \frac{14}{\ln(\frac{1}{4})} \ln(12) + 16 - \frac{14 \ln(8)}{\ln(\frac{1}{4})}$$

$$\approx \boxed{11.91 \text{ miles each hour}}$$

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Ex 3. A substance is being converted into a second substance at a rate proportional to the square of the amount of the first substance present at any time t . Initially, 38 grams of the first substance was present, and 1 hour later 15 grams remained. What is the amount of the first substance remaining after 5 hours? (Round to 2)

$A(t)$ = amount of first substance after t hours

$$\frac{dA}{dt} = kA^2$$

$$\int A^{-2} dA = \int k dt$$

$$-A^{-1} = kt + C$$

$$\frac{1}{A} = -kt + C$$

$$A = \frac{1}{-kt + C}$$

when $t=0$, $A=38$

$$38 = \frac{1}{C}, \text{ so } C = \frac{1}{38}$$

$$A = \frac{1}{-kt + \frac{1}{38}}$$

when $t=1$, $A=15$

$$15 = \frac{1}{-k + \frac{1}{38}}$$

$$\frac{1}{15} = -k + \frac{1}{38}$$

$$-k = \frac{1}{15} - \frac{1}{38} = \frac{23}{570}$$

$$A = \frac{1}{\frac{23}{570}t + \frac{1}{38}}$$

$$A(5) = \frac{1}{\frac{23}{570}(5) + \frac{1}{38}} \approx \boxed{4.38 \text{ grams}}$$

Ex 4. Find the general solution

$$12x^2y' = y' + 3xe^{-y}$$

$$12x^2y' - y' = 3xe^{-y}$$

$$(12x^2 - 1) \frac{dy}{dx} = 3xe^{-y}$$

$$e^y dy = \frac{3x}{12x^2 - 1} dx$$

$$u = 12x^2 - 1$$

$$du = 24x dx$$

$$\frac{1}{8} du = 3x dx$$

$$\int e^y dy = \frac{1}{8} \int \frac{1}{u} du$$

$$e^y = \frac{1}{8} \ln|u| + C$$

$$e^y = \frac{1}{8} \ln|12x^2 - 1| + C$$

$$\boxed{y = \ln\left(\frac{1}{8} \ln|12x^2 - 1| + C\right)}$$