MA 16020

Lesson R(eview)

The Fundamental Theorem of Calculus

Recall that a function $F(x)$ is called an <u>antiderivative</u> of a function $f(x)$ if
$F'(x) = f(x)$.

Each function has infinitely many antiderivatives, and any two antiderivatives differ by a constant. (they change the same way, so they must have the same shape, but could be shifted vertically.)

The <u>indefinite integral</u> of a function $f(x)$ is denoted by $\int f(x)\,dx$ and gives the most general antiderivative of $f(x)$. Generally, it is of the form $F(x) + C$.

<u>Example 1.</u> Find $\int \left(\sqrt[3]{x} + 4\cos x - \frac{5}{x} + e^x\right) dx$

Can split up! $\int x^{1/3}\,dx + 4\int \cos x\,dx - 5\int \frac{1}{x}\,dx + \int e^x\,dx$

$$\frac{3}{4}x^{4/3} + 4\sin x - 5\ln|x| + e^x + C$$

$$\boxed{\frac{3}{4}x^{4/3} + 4\sin x - 5\ln|x| + e^x + C}$$

You can always check by differentiating
Don't forget the $+C$.

The area between a function $f(x)$ and the x-axis from $x = a$ to $x = b$ is called the <u>integral</u> of $f(x)$ from a to b, denoted $\int_a^b f(x)\,dx$.

Area above the x-axis is taken to be positive and below the x-axis is taken to be negative.

MA 16020

Lesson R

Fundamental Theorem of Calculus

If $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x) = F(b) - F(a)$$

Ex 2. Compute $\int_0^1 (x\sqrt[3]{x^2} + x)\,dx$

$$\sqrt[3]{x^2} = x^{2/3}, \text{ so } x\sqrt[3]{x^2} = x \cdot x^{2/3} = x^{5/3}$$

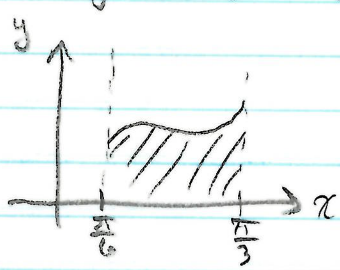
$$\int_0^1 (x^{5/3} + x)\,dx = \left(\tfrac{3}{8}x^{8/3} + \tfrac{1}{2}x^2\right)\Big|_0^1$$

$$= \left(\tfrac{3}{8}(1)^{8/3} + \tfrac{1}{2}(1)^2\right) - \left(\tfrac{3}{8}(0)^{8/3} + \tfrac{1}{2}(0)^2\right)$$

$$= \tfrac{3}{8} + \tfrac{1}{2} = \boxed{\tfrac{7}{8}}$$

Ex 3. Find the area of the region bounded by the graphs of

$$y = 3\sec^2 x + \sin x,\quad y = 0,\quad x = \tfrac{\pi}{6},\quad x = \tfrac{\pi}{3}$$



This area is simply

$$\int_{\pi/6}^{\pi/3} (3\sec^2 x + \sin x)\,dx$$

$$= (3\tan x - \cos x)\Big|_{\pi/6}^{\pi/3}$$

$$= \left(3\tan(\tfrac{\pi}{3}) - \cos(\tfrac{\pi}{3})\right) - \left(3\tan(\tfrac{\pi}{6}) - \cos(\tfrac{\pi}{6})\right)$$

$$= \left(3\sqrt{3} - \tfrac{1}{2}\right) - \left(3\tfrac{\sqrt{3}}{3} - \tfrac{\sqrt{3}}{2}\right)$$

$$= 3\sqrt{3} - \tfrac{1}{2} - \sqrt{3} + \tfrac{\sqrt{3}}{2}$$

$$= \boxed{\frac{5\sqrt{3} - 1}{2}}$$

MA 16020
Lesson R

Ex 4. At 3pm, there is no snow on the ground and snow begins to fall. The rate of change of the amount of snow is given by $A'(t) = 3t+2$ mm/hr, where t is measured in hours after 3pm.

(a) How much snow falls between 4pm and 6pm?

4pm is $t=1$, 6pm is $t=3$

So want $A(3) - A(1) = \int_1^3 A'(t)\,dt$

$$\int_1^3 (3t+2)\,dt = \left(\frac{3}{2}t^2 + 2t\right)\Big|_1^3$$

$$= \left(\frac{3}{2}(3)^2 + 2(3)\right) - \left(\frac{3}{2}(1)^2 + 2(1)\right)$$

$$= \frac{27}{2} + 6 - \frac{3}{2} - 2$$

$$= \boxed{16 \text{ mm}}$$

(b) After how many hours after 3pm will there be 3 cm of snow? Round to the nearest tenth.

3 cm = 30 mm

The amount of snow b hours after 3pm is

$$\int_0^b A'(t)\,dt \overset{\text{want}}{=} 30$$

$$\int_0^b (3t+2)\,dt = \left(\frac{3}{2}t^2 + 2t\right)\Big|_0^b = \frac{3}{2}b^2 + 2b$$

want $\frac{3}{2}b^2 + 2b = 30 \Leftrightarrow \frac{3}{2}b^2 + 2b - 30 = 0$

$$b = \frac{-2 \pm \sqrt{4 - 4\left(\frac{3}{2}\right)(-30)}}{2\left(\frac{3}{2}\right)} = \frac{-2 \pm \sqrt{184}}{3}$$

$b \approx 3.9$ or -5.2

negative answer doesn't make sense

so $\boxed{3.9 \text{ hours}}$