Eddie Price **Inverse of a 2×2 Matrix** **Spring 2018**

Given an arbitrary 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we will see below that $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, if $ad - bc \neq 0$.

We do this by using the augmented matrix, and assuming we can divide by a. (If $a = 0$, then check if $c = 0$. If both a and c are 0, then $ad - bc = 0$, and there is no inverse; but if $a = 0$ and $c \neq 0$, then swap rows 1 and 2 and relabel the entries of the matrix.)

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{a}R_1 \mapsto R_1} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{-cR_1 + R_2 \mapsto R_2} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & d - \frac{cb}{a} & \frac{-c}{a} & 1 \end{array}\right]$$

Now, $d - \frac{cb}{a} = \frac{ad}{a} - \frac{bc}{a} = \frac{ad-bc}{a}$

$$\left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{ad-bc}{a} & \frac{-c}{a} & 1 \end{array}\right] \xrightarrow{\frac{a}{ad-bc}R_2 \mapsto R_2} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\xrightarrow{-\frac{b}{a}R_2 + R_1 \mapsto R_1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{a} + \frac{bc}{a(ad-bc)} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\frac{1}{a} + \frac{bc}{a\,(ad-bc)} = \frac{ad-bc}{a\,(ad-bc)} + \frac{bc}{a\,(ad-bc)} = \frac{ad-bc+bc}{a\,(ad-bc)} = \frac{ad}{a\,(ad-bc)} = \frac{d}{ad-bc}$$

So we get the inverse matrix

$$A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

We can check that $AA^{-1} = A^{-1}A = I_2$:

$$AA^{-1} \longrightarrow \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} \qquad A^{-1}A \longrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \frac{ad-bc}{ad-bc} & \frac{-ab+ba}{ad-bc} \\ \frac{cd-cd}{ad-bc} & \frac{-bc+ad}{ad-bc} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} \begin{bmatrix} \frac{ad-bc}{ad-bc} & \frac{-ab+ba}{ad-bc} \\ \frac{-ac+ac}{ad-bc} & \frac{-bc+ad}{ad-bc} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

<u>Example</u>: find the inverse matrix of $A = \begin{bmatrix} 2 & 2 \\ -1 & 3 \end{bmatrix}$. Notice that $ad - bc = (2)\,(3) - (2)\,(-1) = 6 + 2 = 8$. Hence,

$$A^{-1} = \frac{1}{8}\begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & \frac{-1}{4} \\ \frac{1}{8} & \frac{1}{4} \end{bmatrix}$$