Eddie Price An example where the constraint has an inequality Spring 2018

Problem: Alice has exactly 24 hours to study for an exam, and without preparation, she will score 400 out of 1000 points. Her exam score will improve by $x\,(50-x)$ points for x hours of reading her notes and will improve by $y\,(60-y)$ points for y hours of doing practice problems. She will lose $(x+y)^2$ points from fatigue. What is the maximum score she can obtain, if she allows herself the option of sleeping for part of this 24 hour period?

On your homework tonight, you will assume that Alice is not allowed to sleep at all, and will solve that problem subject to the constraint $x+y=24$. But here, we will consider the case where Alice can sleep, giving us the constraint $x+y\le 24$.

The first thing we do is set up a function for the number of points Alice will earn:

$$P(x,y) = 400 + x\,(50-x) + y\,(60-y) - (x+y)^2$$
$$P(x,y) = 400 + 50x - 2x^2 + 60y - 2y^2 - 2xy$$

Here, we have the constraint $x+y\le 24$. We can use Lagrange multipliers to find candidate points for the maximum living on the boundary, $x+y=24$.

$$P_x = 50-4x-2y, \qquad P_y = 60-4y-2x, \qquad g_x = 1, \qquad g_y = 1$$

We get the system of equations:

$$50-4x-2y=\lambda, \qquad 60-4y-2x=\lambda, x+y=24$$

Seeing that the first two equations are both solved for λ, we get $50-4x-2y=60-4y-2x$, which, after doing some algebra, is $x-y=-5$. We also have the equation $x+y=24$. Solving this system of equations, we get $x=\frac{19}{2}$ and $y=\frac{29}{2}$.

But this is a candidate point *on* the boundary of our region of interest. We must also observe the critical points of $P(x,y)$. We find these by setting both partial derivatives $P_x=0$ and $P_y=0$, so we want to satisfy the system of equations

$$50-4x-2y=0, \qquad 60-4y-2x=0$$

We can find that the solution is $x=\frac{20}{3}$ and $y=\frac{35}{3}$. So $\left(\frac{20}{3},\frac{35}{3}\right)$ is a critical point of P. We must check that this point actually satisfies the constraint, $x+y\le 24$. But $\frac{20}{3}+\frac{35}{3}=\frac{55}{3}\approx 18.333<24$, so it does satisfy the constraint.

Now, we check the two candidate points we found.

$$P\left(\frac{19}{2},\frac{29}{2}\right)\approx 869, \qquad P\left(\frac{20}{3},\frac{35}{3}\right)\approx 917$$

So the maximum score Alice can get is 917 points if she reads her notes for $\frac{20}{3}$ hours $\approx$ 6 hours and 40 minutes, does practice problems for $\frac{35}{3}$ hours $\approx$ 11 hours and 20 minutes, and sleeps for approximately 5 hours and 40 minutes. But this is only because she is allowing herself to sleep. If she did not allow herself to sleep, then the maximum score would be 869 points.