Eddie Price What's going on with constrained optimization? Spring 2018

To understand what's going on with constrained optimizations in MA 16020, let's look at how constrained optimization worked in single variable calculus. Let $y = f(x)$ and suppose you want to find the maximum value of f on the interval $[a, b]$. By restricting to the interval $[a, b]$, you're actually placing a constraint on your function. You only care about x-values satisfying $a \leq x \leq b$. To find the maximum value, you must check critical numbers between a and b as well as the endpoints a and b themselves! Since we are imposing our own domain restrictions for non-mathematical reasons, it is possible that the maximum value occurs at an endpoint even if that endpoint is not a critical number.

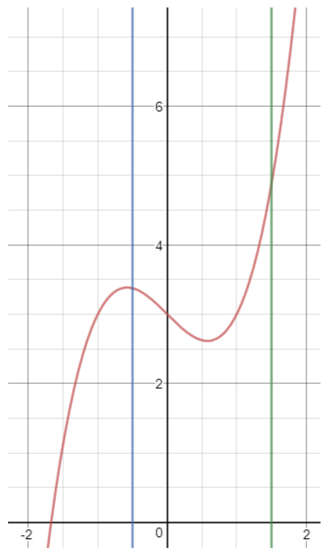

Figure 1: created in Desmos

In the above picture, notice that if we restict our domain to the interval between the two vertical lines, our maximum value occurs at the right endpoint, even though the function doesn't have a relative maximum there.

The same thing happens when we are putting a constraint $g(x, y) = C$ on a multivariate function. We're restricting our domain to a specific curve in the plane.

For example, suppose we have a function $f(x, y)$ and we want to minimize it subject to the constraint $x^2 + y^2 = 4$. Normally, $f(x, y)$ is a surface living above and below the xy-plane. But subjecting it to the constraint $x^2 + y^2 = 4$, we are only restricting our attention to the part of the surface living above the circle $x^2 + y^2 = 4$. Much like in the single variable case, the minimum for the constrained domain could occur at an "endpoint" (above the circle in the plane) even if the function doesn't have a critical point on that circle.

The Method of Lagrange Multipliers allows us to find candidate points along that boundary curve for where the function could have a max or min value.

But what if you have a constraint like $g(x, y) \leq C$? e.g., $x^2 + y^2 \leq 4$?

This constraint restricts our domain to everything on and inside the circle. The Method of Lagrange Multipliers only helps us find candidate points on the boundary of the circle itself, not inside the circle. In such a case, we should check all critical points inside the circle as well. It's possible that the global max/min occurs *inside* the boundary or *on* it.

Looking back at our picture from the single variable case, we can see that the global minimum of the function between the two horizontal lines occurs at a genuine critical point of the function.

So for the multivariate case, you would use Lagrange multipliers to find candidate extrema *on* the boundary $x^2+y^2 = 4$, and then find all critical points (a, b) (as you did for the second derivative test) which satisfy $a^2 + b^2 \leq 4$. You would then check the values of all of these critical points and candidate points.