Eddie Price **Sum of a Geometric Series** **Spring 2018**

Consider the geometric series $a+ar+ar^2+ar^3+...+ar^n+...$ with first term a and common ratio r. We know that if $-1<r<1$, then this has the sum $\frac{a}{1-r}$. In class, I gave you the correct way to see that this works, by investigating the limit of the partial sums. But there is another way to see that if the series converges, then its sum must be $\frac{a}{1-r}$.

Let $S=a+ar+ar^2+ar^3+...+ar^n+...$. If you assume that the sum of the series makes sense and that you can use normal algebra rules on it, then the following argument makes sense:

Multiply both sides by r:

$$rS=r\left(a+ar+ar^2+ar^3+...+ar^n+...\right)=ar+ar^2+ar^3+ar^4+...+ar^n+...$$

Since there are infinitely many terms, we see that the only difference between this series and our original series S is a. In other words,

$$a+rS=S$$

We want to solve for S, so we subtract rS from both sides:

$$a=S-rS$$

Factoring out S, we get

$$a=(1-r)\,S$$

Dividing both sides by $1-r$, we get

$$S=\frac{a}{1-r}$$

So this shows that ***if*** the series has a sum, ***then*** the sum is $\frac{a}{1-r}$. But we can't tell from this method when the series actually has a sum.

(For example, if $a=1$ and $r=2$, then the series is $1+2+4+8+16+...$, which clearly diverges to ∞, and does *not* have the value $\frac{a}{1-r}=\frac{1}{1-2}=-1$.)

This is why we needed to look at the limit of partial sums in class.

(See next page for an explanation on partial sums.)

In class, I told you that the nth partial sum of the geometric series $a + ar + ar^2 + ar^3 + ... + ar^n + ...$ is $a \cdot \frac{1-r^{n+1}}{1-r}$ (provided that $r \neq 1$). I will now show you why this is true.

Recall, the nth partial sum is simply the sum of the first n terms: $a+ar+ar^2+ar^3+...+ar^n$. We will let S_n denote this sum. Now, we can use the same sort of argument as on the previous page! (Because S_n definitely makes sense being the sum of finitely many terms.)

Multiply both sides by r to get

$$rS_n = r\left(a + ar + ar^2 + ar^3 + ... + ar^n\right) = ar + ar^2 + ar^3 + ar^4 + ... + ar^n + ar^{n+1}$$

Now, since there are only finitely many terms here, we have a bit of a difference from before. In order to get our original sum S_n, we have to add a and subtract ar^{n+1}.

$$a + rS_n - ar^{n+1} = S_n$$

Since we want to solve for S_n, we subtract rS_n from both sides to get

$$a - ar^{n+1} = S_n - rS_n$$

We factor out a on the left hand side and S on the right hand side:

$$a\left(1 - r^{n+1}\right) = (1 - r)\,S_n$$

Now, we divide both sides by $1 - r$ to get

$$S_n = a \cdot \frac{1 - r^{n+1}}{1 - r}$$

This gives the partial sum formula I described in class. And of course, analyzing the limit as $n \to \infty$, we see that we get $\lim_{n\to\infty} r^{n+1} = 0$ whenever $-1 < r < 1$, and the whole formula diverges whenever $r \leq -1$ or $r \geq 1$.