Here's a brief explanation for why the integrating factor method works. Given a first order linear differential equation

$$\frac{dy}{dt} + P(t)\,y = Q(t)$$

We notice that the left hand side of the equation *almost* looks like the result of the product rule where one of the factors is y. Given some function $u(t)$ (which we will also denote by u), $\frac{d}{dt}\,[uy] = u(t)\,\frac{dy}{dt} + u'(t)\,y$, which looks an awful lot like the left hand side of the differential equation we started with.

We definitely notice a difference, though. Our original equation doesn't have $\frac{dy}{dt}$ multiplied by any function. So if we multiply our differential equation by $u(t)$ (to try to get them to match), we get

$$u(t)\,\frac{dy}{dt} + u(t)\,P(t)\,y = u(t)\,Q(t)$$

Now, the left hand side is much closer to the result of the product rule for finding the derivative of uy. The only thing we need to be true is that $u'(t)$ needs to be equal to $u(t)\,P(t)$. This gives the separable differential equation $\frac{du}{dt} = u \cdot P(t)$, which has the function $u(t) = e^{\int P(t)\,dt}$ as a solution.

<u>Main Point</u>: Choosing $u(t) = e^{\int P(t)\,dt}$ is specifically what we need to do in order to have the following situation: When we multiply the differential equation $\frac{dy}{dt} + P(t)\,y = Q(t)$ by $u(t)$, the left hand side will be equal to the result of taking the derivative of uy.

As such, when we multiply by $u(t)$ (based on how $u(t)$ is defined), we get

$$\frac{d}{dt}\,[uy] = u(t)\,Q(t)$$

This makes it possible to solve for y since the integral of the derivative of a function is that function itself (up to an arbitrary constant, which we don't have to worry about since we can put it on the other side). So now, we can integrate both sides to get

$$\int \frac{d}{dt}\,[uy]\ dt = \int u(t)\,Q(t)\ dt$$

$$u(t)\,y = \int u(t)\,Q(t)\ dt + C$$

And then we can solve for y easily.