MA 16020
Lesson 10
First order Linear Diff Eqs (Part 2)

Ex 1. A pastry shop has capacity to hold 100 pastries. There are currently 65 pastries in the shop, and they are selling pastries at an hourly rate of 25% of the avialable capacity. When will they sell out? (Round to 2 decimals)

Let $A(t)$ = amount of pastries in the shop.

$$\frac{dA}{dt} = \underbrace{-}_{\text{amount of pastries is decreasing}}\underbrace{0.25}_{25\%}\underbrace{(100 - A)}_{\text{available capacity}}$$

$$\frac{dA}{dt} = -25 + 0.25A, \text{ so } \star\ \frac{dA}{dt} - 0.25A = -25\ \star$$

$$u(t) = e^{\int -0.25\,dt} = e^{-0.25t}$$

$$\frac{d}{dt}\left[e^{-0.25t}A\right] = -25e^{-0.25t}$$

Need $\int -25e^{-0.25t}\,dt$, $w = -0.25t$, $dw = -0.25\,dt$

$$\frac{-25}{-0.25}\int e^{w}\,dw = 100e^{w} + C = 100e^{-0.25t} + C$$

$$e^{-0.25t}A = 100e^{-0.25t} + C$$

$$A(t) = 100 + Ce^{0.25t}$$

when $t = 0$, $A = 65$

$$65 = 100 + C, \quad C = -35$$

$$A(t) = 100 - 35e^{0.25t}$$

$$0 = 100 - 35e^{0.25t}$$

$$\frac{100}{35} = e^{0.25t}$$

$$\ln\left(\frac{100}{35}\right) = 0.25t$$

$$t = \frac{\ln\left(\frac{100}{35}\right)}{0.25} \approx \boxed{4.20 \text{ hours}}$$

Ex 2. A 600 gallon tank initially contains 300 gallons of brine containing 50 lbs of salt. Brine containing 4 lbs of salt/gal flows into the tank at a rate of 5 gal/min, and the well-stirred mixture flows out at a rate of 1 gal/min. How much salt is in the tank when it is full? (Round to 2 decimals)

Notice: Volume increases by $5-1=4$ gal/min, so
$V(t)=300+4t$

$$\frac{dA}{dt}=\left(\frac{4\text{ lbs}}{\text{gal}}\right)\left(\frac{5\text{ gal}}{\text{min}}\right)-\left(\frac{A\text{ lbs}}{300+4t\text{ gal}}\right)\left(\frac{1\text{ gal}}{\text{min}}\right)$$
$$=20-\frac{1}{300+4t}A$$

So $\frac{dA}{dt}+\frac{1}{300+4t}A=20$

$u(t)=e^{\int\frac{1}{300+4t}dt}$, $\int\frac{1}{300+4t}dt$, $w=300+4t$, $dw=4\,dt$, $\frac{1}{4}dw=dt$

$=\frac{1}{4}\int\frac{1}{w}dw=\frac{1}{4}\ln|w|=\frac{1}{4}\ln|300+4t|$

So $u(t)=e^{\ln(300+4t)^{1/4}}=(300+4t)^{1/4}$

$(300+4t)^{1/4}A=\int 20(300+4t)^{1/4}dt$, $w=300+4t$, $\frac{1}{4}dw=dt$

$=5\int w^{1/4}dw=5\cdot\frac{4}{5}w^{5/4}+C$

$(300+4t)^{1/4}A=4(300+4t)^{5/4}+C$

$A(t)=4(300+4t)+C(300+4t)^{-1/4}$

when $t=0$, $A=50$

$50=4(300)+C(300)^{-1/4}$, so $C=-1150(300)^{1/4}$

$A(t)=4(300+4t)-1150(300)^{1/4}(300+4t)^{-1/4}$

Tank is full when $600=V(t)=300+4t$, $t=75$

$A(75)=4(600)-1150(300)^{1/4}(600)^{-1/4}$

$\approx\boxed{1432.97\text{ lbs}}$

Ex 3. A 20,000 ft^3 room has 900 picocuries of radon per ft^3. 700 ft^3 of air per hour flows into the room containing 6 picocuries per ft^3. The well-mixed air leaves the room at the same rate. If 98 picocuries per ft^3 is a safe breathing level, how long will it take until it is safe to breathe? (2 decimals)

$A(t)$ = picocuries of radon in the room

$$\frac{dA}{dt} = \left(\frac{6 \text{ picocuries}}{ft^3}\right)\left(\frac{700\ ft^3}{h}\right) - \left(\frac{A \text{ picocuries}}{20{,}000\ ft^3}\right)\left(\frac{700\ ft^3}{h}\right)$$

$$\frac{dA}{dt} = 4200 - \frac{7}{200}A, \text{ so } \star\ \frac{dA}{dt} + \frac{7}{200}A = 4200\ \star$$

$$u(t) = e^{\int \frac{7}{200}dt} = e^{7t/200}$$

$$\frac{d}{dt}\left[e^{7t/200}A\right] = 4200e^{7t/200}$$

$$\int 4200e^{7t/200}dt, \quad w = \frac{7t}{200}, \quad dw = \frac{7}{200}dt, \quad \frac{200}{7}dw = dt$$

$$\frac{200}{7}\cdot 4200\int e^w dw = 120{,}000\, e^{7t/200} + C$$

$$e^{7t/200}A = 120{,}000e^{7t/200} + C$$

Can find C now... when $t=0$, $A = \left(\frac{900 \text{ picocuries}}{ft^3}\right)(20{,}000\ ft^3) = 18{,}000{,}000$

$$18{,}00{,}000 = 120{,}000 + C, \quad C = 17{,}880{,}000$$

$$A(t) = 120{,}000 + 17{,}880{,}000e^{-7t/200}$$

Want $\left(\frac{98 \text{ picocuries}}{ft^3}\right)(20{,}000\ ft^3) = 1{,}960{,}000$ picocuries

$$1{,}960{,}000 = 120{,}000 + 17{,}880{,}000e^{-7t/200}$$

$$\frac{1{,}840{,}000}{17{,}880{,}000} = e^{-7t/200}$$

$$\ln\left(\frac{184}{1788}\right) = \frac{-7}{200}t$$

$$t = -\frac{200}{7}\cdot\ln\left(\frac{184}{1788}\right) \approx \boxed{64.97 \text{ hours}}$$

Ex 4. A water-based solution contains 15 grams of undissolved chemicals. The rate of change of the amount of chemicals dissolved in the solution is proportional to the amount of undissolved chemicals. Let k be the positive proportionality constant. Set up a diff eq.

Let Q(t) be the amount of dissolved chemicals.
The amount of undissolved chemicals at time t is $15 - Q$.

$$\frac{dQ}{dt} = \underbrace{k}_{\text{positive}}\underbrace{(15 - Q)}_{\substack{\text{positive} \\ (\text{since } Q \leq 15)}}$$

and Q is increasing, so $\frac{dQ}{dt}$ should be positive

$$\frac{dQ}{dt} = k(15 - Q)$$