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Lesson 11

Area Between Curves

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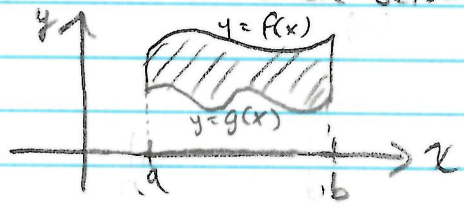
Recall that you can find the area between a curve $y=f(x)$ and the x-axis from $x=a$ to $x=b$ by taking $\int_a^b f(x)\,dx$.

What if you want the area between two curves?



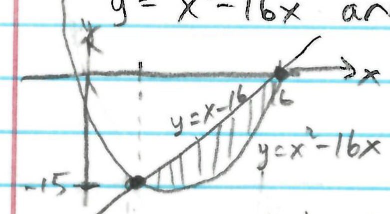

Notice, you can take area below $f(x)$ and subtract area below $g(x)$

so $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b (f(x)-g(x))\,dx$

To find the area bounded by curves,

1. Sketch a somewhat accurate plot of the curves/regions
2. Use the sketch/algebra to determine the bounds of integration and the upper curve and lower curve (or right curve and left curve)
3. $\int_a^b (\text{upper}-\text{lower})\,dx$ or $\int_a^b (\text{right}-\text{left})\,dy$

Ex 1. Find the area of the region bounded by $y = x^2-16x$ and $y = x-16$




intersect

$x^2-16x = x-16$

$x^2-17x+16=0$

$(x-1)(x-16)=0 \Rightarrow x=1,\ x=16$

top curve: $x-16$, bottom curve: $x^2-16x$

$$\int_1^{16} [(x-16)-(x^2-16x)]\,dx = \int_1^{16} (17x-16-x^2)\,dx$$

$$= \left(\tfrac{17}{2}x^2-16x-\tfrac{1}{3}x^3\right)\Big|_1^{16}$$

$$= \left[\tfrac{17}{2}(16)^2-16(16)-\tfrac{1}{3}(16)^3\right]-\left[\tfrac{17}{2}(1)^2-16(1)-\tfrac{1}{3}(1)^3\right]$$

$$= \boxed{562.5}$$

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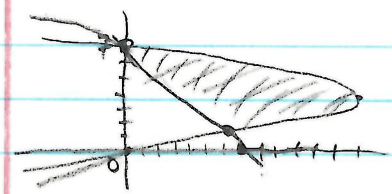
Ex 2. Find the area bounded by the curves
$\underbrace{x = 7y - y^2}_{\text{parabola opening left }\supset}$ and $\underbrace{x + y = 7}_{\text{line}}$



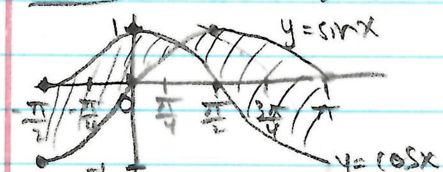

right curve: $x = 7y - y^2$
left curve: $x = -y + 7$

Set equal to find intersection points:

$$7y - y^2 = -y + 7$$
$$-y^2 + 8y - 7 = 0 \Rightarrow -(y-7)(y-1) = 0$$
$$y = 1, \quad y = 7$$

$$\int_1^7 \left[(7y - y^2) - (-y + 7)\right] dy = \int_1^7 (8y - y^2 - 7)\, dy$$
$$= \left(4y^2 - \tfrac{1}{3}y^3 - 7y\right)\Big|_1^7 = \left[4(7)^2 - \tfrac{1}{3}(7)^3 - 7(7)\right] - \left[4(1)^2 - \tfrac{1}{3}(1)^3 - 7(1)\right]$$
$$= \boxed{36}$$

Ex 3. Find the area bounded by $y = \sin x$, $y = \cos x$, $x = -\frac{\pi}{2}$, $x = \pi$

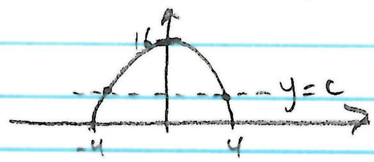



from $-\frac{\pi}{2}$ to $\frac{\pi}{4}$, cosx top, sinx bottom

from $\frac{\pi}{4}$ to $\pi$, sinx top, cosx bottom

$$\int_{-\pi/2}^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi} (\sin x - \cos x)\, dx$$
$$= (\sin x + \cos x)\Big|_{-\pi/2}^{\pi/4} + (-\cos x - \sin x)\Big|_{\pi/4}^{\pi}$$
$$= \left[\sin(\tfrac{\pi}{4}) + \cos(\tfrac{\pi}{4})\right] - \left[\sin(-\tfrac{\pi}{2}) + \cos(-\tfrac{\pi}{2})\right] + \left[-\cos(\pi) - \sin(\pi)\right] - \left[-\cos(\tfrac{\pi}{4}) - \sin(\tfrac{\pi}{4})\right]$$
$$= \left[\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}\right] - \left[-1 + 0\right] + \left[1 - 0\right] - \left[-\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2}\right]$$
$$= \sqrt{2} + 1 + 1 + \sqrt{2}$$
$$= \boxed{2\sqrt{2} + 2}$$

Ex 4. Find the equation of the horizontal line that divides the area of the region in half: $y = 16 - x^2$, $y = 0$




Find actual area

$$\int_{-4}^{4} (16 - x^2)\,dx = 2\int_0^4 (16 - x^2)\,dx$$

(since symmetric)

$$= 2\left(16x - \tfrac{1}{3}x^3\right)\Big|_0^4$$

$$= 2\left[16(4) - \tfrac{1}{3}(4)^3 - 0\right] = \frac{256}{3}$$

So half area is: $\frac{256}{3} \cdot \frac{1}{2} = \frac{128}{3}$

Find area of upper half: can go right to left:

$y = 16 - x^2 \Rightarrow x^2 = 16 - y \Rightarrow x = \pm\sqrt{16 - y}$

$$\int_c^{16} \left(\sqrt{16-y}\right) - \left(-\sqrt{16-y}\right) dy = \int_c^{16} 2\sqrt{16-y}\,dy$$

$u = 16 - y$, $du = -dy$, $-du = dy$

$$-2\int_{16-c}^{0} u^{1/2}\,du = -2 \cdot \tfrac{2}{3} u^{3/2}\Big|_{16-c}^{0}$$

$$= \left[\left(-\tfrac{4}{3}\right)(0)^{3/2}\right] - \left[-\tfrac{4}{3}(16-c)^{3/2}\right]$$

$$= \tfrac{4}{3}(16-c)^{3/2}$$

Want: $\frac{4}{3}(16-c)^{3/2} = \frac{128}{3}$

$$(16-c)^{3/2} = 32$$

$$16 - c = (32)^{2/3}$$

$$c = 16 - (32)^{2/3}$$

$$\boxed{y = 16 - (32)^{2/3}}$$

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Remember: $\int_a^b f(x)\,dx$ represents the net change of $F(x)$ from $x=a$ to $x=b$

Also, Profit = Revenue - cost