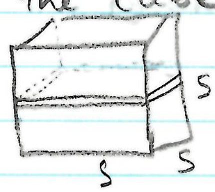
MA 16020

Lesson 12

Volumes of Solids of Revolution - Disks

Volumes of a solid are deeply related to the areas of cross-sections of those solids.

Consider the Cube

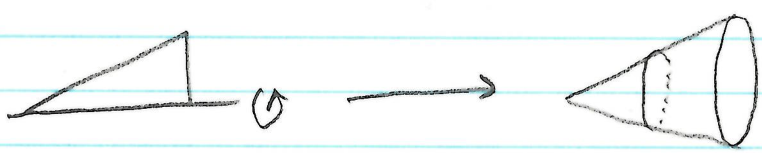


Taking a cross section of height Δx, we get lots of small volumes of $s^2 \Delta x$.

Adding up all of these cross-sections gives a Riemann sum approximating the volume.

Taking the limit thus gives the actual volume, but it also gives an integral $\int_0^s s^2\,dx$ where the integrand is the cross sectional area.

In the next three lessons, we rotate a region around a line to create a solid



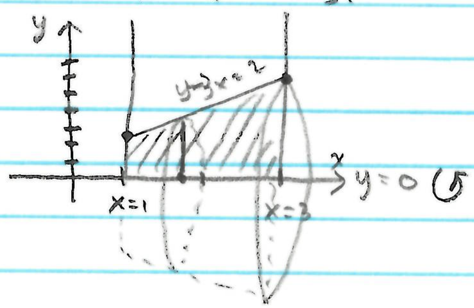
When rotating, the cross-sections are circular, and thus have an area of πr^2

Disk Method

If you rotate a region around a line and there are no gaps, then cross-sections are disks!

1. Sketch the region and a sample disk of the solid
2. If you rotate around x-axis, get all relevant functions in terms of x; similarly, for y-axis, get in terms of y
3. Determine the radius of the disks (a function) and where the disks start and stop
4. $V = \int_a^b \pi r^2 (dx \text{ or } dy)$

Ex 1. Find the volume of the solid obtained by rotating the region bounded by $y-3x=2$, $x=1$, $x=3$, $y=0$ around the x-axis.



$y-3x=2$ is a line

rotate about x-axis
get all functions in terms of x
$y=3x+2$

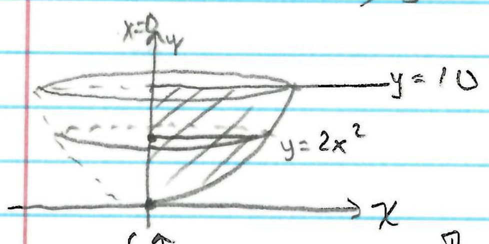
The radius of the disk is $y=3x+2$. x varies from 1 to 3.

$$V=\int_1^3 \pi(3x+2)^2\,dx = \pi\int_1^3 (9x^2+12x+4)\,dx$$

$$=\pi\left[3x^3+6x^2+4x\right]\Big|_1^3 = \pi\left[\left[3(3)^3+6(3)^2+4(3)\right]-\left[3(1)^3+6(1)^2+4(1)\right]\right]$$

$$=\pi(134) = \boxed{134\pi}$$

Ex 2. Find the volume of the solid obtained by rotating around the y-axis the region in the first quadrant bounded by $y=2x^2$, $x=0$, $y=10$



rotate about y-axis:
get all functions in terms of y
$y=2x^2 \Rightarrow \frac{1}{2}y=x^2 \Rightarrow x=\pm\sqrt{\frac{1}{2}y}$

The radius of the disk is $x=\sqrt{\frac{1}{2}y}$

y varies from 0 to 10

$$V=\int_0^{10}\pi\left(\sqrt{\tfrac{1}{2}y}\right)^2dy = \pi\int_0^{10}\tfrac{1}{2}y\,dy = \pi\left[\tfrac{1}{4}y^2\right]\Big|_0^{10}$$

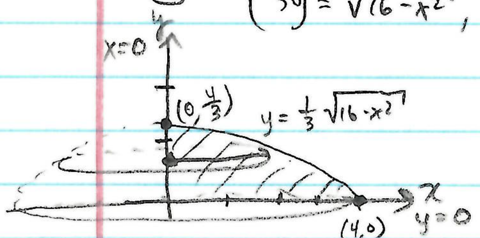
$$=\pi\left[\tfrac{1}{4}(10)^2-\tfrac{1}{4}(0)^2\right] = \boxed{25\pi}$$

MA 16020
Lesson 12

Ex 3. Find the volume of the solid obtained by rotating the region bounded by $y = \frac{1}{3}\sqrt{16-x^2}$, $y=0$, and $x=0$ about the y-axis.

$y = \frac{1}{3}\sqrt{16-x^2}$ is part of an ellipse

$\left(3y = \sqrt{16-x^2}, \text{ so } 9y^2 = 16-x^2, \text{ so } 9y^2 + x^2 = 16\right)$



around y-axis: get everything in terms of y

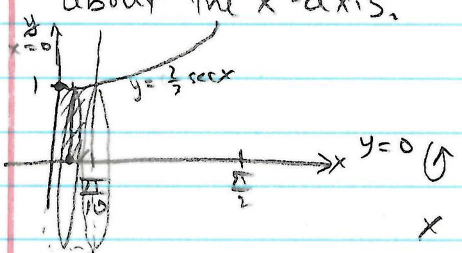
$x^2 = 16 - 9y^2 \Rightarrow x = \pm\sqrt{16-9y^2}$

radius of disk is $x = \sqrt{16-9y^2}$

y varies from 0 to $\frac{4}{3}$

$$V = \int_0^{4/3} \pi\left(\sqrt{16-9y^2}\right)^2 dy = \pi\int_0^{4/3}(16-9y^2)\,dy$$
$$= \pi(16y - 3y^3)\Big|_0^{4/3} = \pi\left[16(4/3) - 3(4/3)^3 - 0 + 0\right]$$
$$= \boxed{\frac{128\pi}{9}}$$

Ex 4. Find the volume of the solid obtained by rotating the region bounded by $y = \frac{2}{3}\sec(x)$, $y=0$, $x=0$, $x=\frac{\pi}{10}$ about the x-axis.



around x-axis: in terms of x

$y = \frac{2}{3}\sec(x)$.

radius of disk is $y = \frac{2}{3}\sec(x)$.

x varies from 0 to $\frac{\pi}{10}$

$$V = \int_0^{\pi/10} \pi\left(\frac{2}{3}\sec x\right)^2 dx = \frac{4\pi}{9}\int_0^{\pi/10} \sec^2 x\,dx$$
$$= \frac{4\pi}{9}\left[\tan x\right]_0^{\pi/10}$$
$$= \frac{4\pi}{9}\left[\tan\left(\frac{\pi}{10}\right) - \tan(0)\right]$$
$$= \boxed{\frac{4\pi}{9}\tan\left(\frac{\pi}{10}\right)}$$