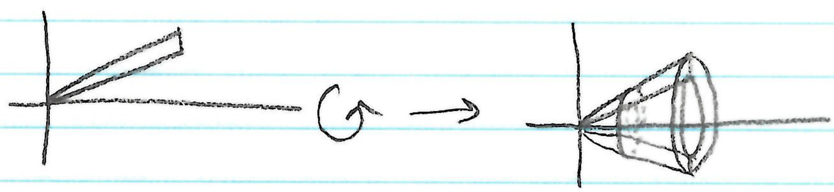
MA 16020

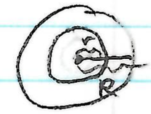
Lesson 13

Volumes of Solids of Revolution (washers)

What if your solid has a gap in it?



now, cross-sections look like washers



If R is the outer radius and r is the inner radius, the area of the washer is $\pi R^2 - \pi r^2 = \pi(R^2 - r^2)$

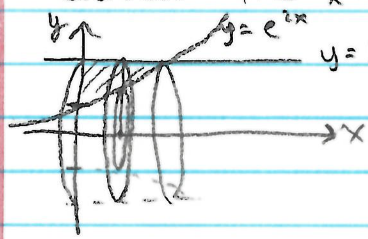
Washer Method

If you rotate a region about a line and the resulting solid has a gap in it, cross-sections are washers.

1. Sketch the region, solid, and sample washer
2. If rotating about x-axis, get all functions in terms of x (if about y-axis, in terms of y)
3. Determine the outer radius R and inner radius r of the sample washer and what x- or y-values the washers vary over.
4. $V = \int_a^b (\pi R^2 - \pi r^2)\, d(x \text{ or } y) = \pi \int_a^b (R^2 - r^2)\, d(x \text{ or } y)$

Notice: The Disk Method is a special case of the Washer Method where the gap has 0 volume (equivalently $r = 0$).

Ex 1. Find the volume of the solid obtained by rotating the region bounded by $y = e^{2x}$, $x = 0$, $y = 3e$ about the x-axis.



$R = 3e$, $r = e^{2x}$

washers vary from $x = 0$ to $x = ?$

where does $3e = e^{2x}$?

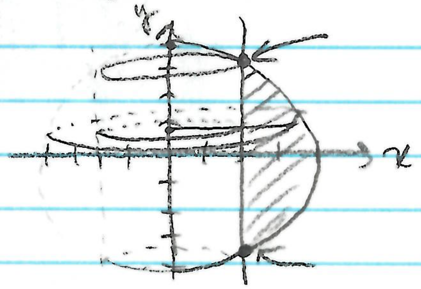
$3 = e^{2x-1} \Rightarrow \ln 3 = 2x - 1 \Rightarrow x = \frac{\ln(3)+1}{2}$

$$V = \pi \int_0^{\frac{\ln(3)+1}{2}} \left[(3e)^2 - (e^{2x})^2\right] dx = \pi \int_0^{\frac{\ln(3)+1}{2}} (9e^2 - e^{4x})\,dx$$

$$= \pi \left[9e^2 x - \frac{1}{4}e^{4x}\right]_0^{\frac{\ln(3)+1}{2}} = \pi\left[9e^2\left(\frac{\ln(3)+1}{2}\right) - \frac{1}{4}e^{2\ln(3)+2} - 0 + \frac{1}{4}\right]$$

$$\approx \boxed{167.78}$$

Ex 2. Find the volume of the solid obtained by rotating the region inside the circle $x^2 + y^2 = 16$ and to the right of the line $x = 2$ about the y-axis.



$x^2 + y^2 = 16 \Rightarrow x^2 = 16 - y^2 \Rightarrow x = \pm\sqrt{16 - y^2}$

$R = \sqrt{16 - y^2}$, $r = 2$

need to find intersection points of circle and line:

$\sqrt{16 - y^2} = 2$

$16 - y^2 = 4$

$y^2 = 12 \Rightarrow y = \pm\sqrt{12}$

washers vary from $y = -\sqrt{12}$ to $y = \sqrt{12}$

$$V = \pi \int_{-\sqrt{12}}^{\sqrt{12}} \left(\left(\sqrt{16 - y^2}\right)^2 - (2)^2\right) dy = \pi \int_{-\sqrt{12}}^{\sqrt{12}} (16 - y^2 - 4)\,dy$$

$$= \pi\left[12y - \frac{1}{3}y^3\right]_{-\sqrt{12}}^{\sqrt{12}} \approx \boxed{174.12}$$

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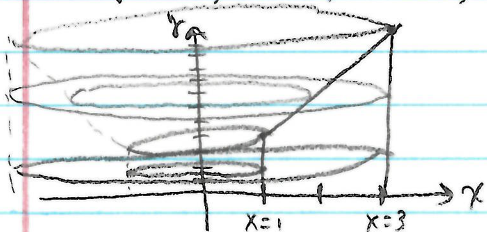
Ex 3. Find the volume of the solid that results by revolving the region enclosed by the curves $y = 4x$, $x = 1$, $x = 3$, and $y = 0$ about the y-axis.

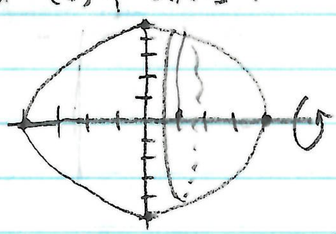


Notice: the solid has 2 parts here (inner radius curve changes!)
From $y=0$ to $y=4$, $R=3$, $r=1$
From $y=4$ to $y=12$, $R=3$, $r=\frac{1}{4}y$

$$V = \pi\int_0^4 \left((3)^2-(1)^2\right)dy + \pi\int_4^{12}\left((3)^2-\left(\tfrac{1}{4}y\right)^2\right)dy$$
$$= \pi\int_0^4 8\,dy + \pi\int_4^{12}\left(9-\tfrac{1}{16}y^2\right)dy$$
$$= \pi\left[8y\right]_0^4 + \pi\left[9y-\tfrac{1}{48}y^3\right]_4^{12} \approx \boxed{217.82}$$

Ex 4. $\frac{x^2}{16}+\frac{y^2}{25}=1$ describes an ellipse. Find the volumes obtained by rotating it around the (a) x-axis and (b) y-axis.

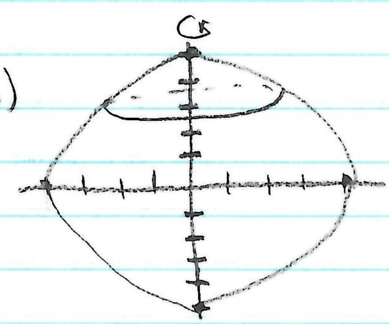
(a)



$$\frac{y^2}{25} = 1-\frac{x^2}{16} \Rightarrow y^2 = 25-\frac{25}{16}x^2$$
$$\Rightarrow y = \pm\sqrt{25-\tfrac{25}{16}x^2}$$

$$V = \pi\int_{-4}^{4}\left(\sqrt{25-\tfrac{25}{16}x^2}\right)^2dx$$
$$= \pi\left[25x-\tfrac{25}{48}x^3\right]_{-4}^{4} \approx \boxed{418.88}$$

(b)



$$\frac{x^2}{16} = 1-\frac{y^2}{25} \Rightarrow x^2 = 16-\frac{16}{25}y^2$$
$$\Rightarrow x = \pm\sqrt{16-\tfrac{16}{25}y^2}$$

$$V = \pi\int_{-5}^{5}\left(\sqrt{16-\tfrac{16}{25}y^2}\right)^2dy$$
$$= \pi\left[16y-\tfrac{16}{75}y^3\right]_{-5}^{5} \approx \boxed{335.10}$$