MA 16020
Lesson 14
Volumes of Solids of Revolution – Other Axes

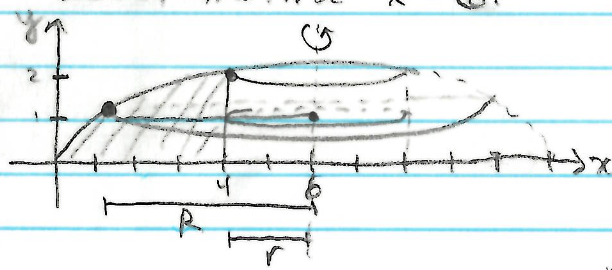
Rotating about axes other than the x- and y-axes generally follows the same procedure.

When rotating about $x=c$ (parallel to y-axis), use functions of y / y-values

For $y=c$ (parallel to x-axis), use functions of x / x-values

The radius or radii are affected by the axis of rotation. Use your picture carefully.

Ex 1. Find the volume of the solid obtained by rotating the region bounded by $y=\sqrt{x}$, $y=0$, and $x=4$ about the line $x=6$.



In terms of y

$x=y^2$

R goes from $x=y^2$ to $x=6$

$R=6-y^2$

r goes from $x=4$ to $x=6$

$r=6-4=2$

washers vary from $y=0$ to $y=2$

$$V=\pi\int_0^2\left[(6-y^2)^2-(2)^2\right]dy$$

$$=\pi\int_0^2(36-12y^2+y^4-4)\,dy$$

$$=\pi\int_0^2(y^4-12y^2+32)\,dy$$

$$=\pi\left[\tfrac{1}{5}y^5-4y^3+32y\right]_0^2=\boxed{\frac{192\pi}{5}}$$

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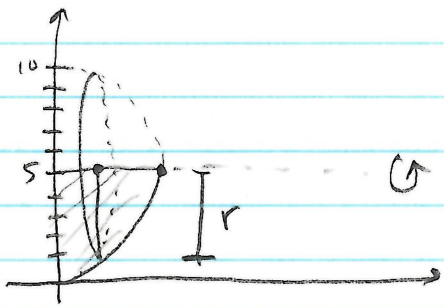
Ex 2. Find the volume of the solid obtained by rotating the region bounded by $x=0$, $y=x^2$, and $y=5$ about the line $y=5$. Round to 3 decimals.



In terms of x

radius goes from $y=x^2$ to $y=5$

$r = 5-x^2$

$x^2=5$ when $x=\pm\sqrt{5}$

disks vary from $x=0$ to $x=\sqrt{5}$

$$V = \pi\int_0^{\sqrt{5}} (5-x^2)^2\,dx = \pi\int_0^{\sqrt{5}} (25-10x^2+x^4)\,dx$$

$$= \pi\left[25x - \tfrac{10}{3}x^3 + \tfrac{1}{5}x^5\right]_0^{\sqrt{5}} \approx \boxed{93.664}$$

problem removed over the past year or so

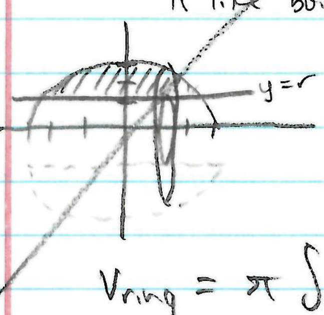
Ex 3. A machinist makes rings by drilling a hole of radius r inches through a sphere of radius 3 inches. What value of r will give the ring a volume of 50% the original sphere's volume?

A sphere of radius 3 can be obtained by rotating upper half of $x^2+y^2=9$ about x-axis:

Sphere has volume $\pi\int_{-3}^{3}\left(\sqrt{9-x^2}\right)^2 dx$

$= \frac{4}{3}\pi(3)^3 = 36\pi$

drilling a hole through center of radius r is like bounding the circle above $y=r$ and rotating about x-axis



$R=\sqrt{9-x^2}$, $r=r$

$9-x^2=r^2$ when $x^2=9-r^2 \Rightarrow x=\pm\sqrt{9-r^2}$

so washers vary from $x=-\sqrt{9-r^2}$ to $\sqrt{9-r^2}$

$$V_{ring} = \pi\int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}}\left[\left(\sqrt{9-x^2}\right)^2 - (r)^2\right]dx$$

$$= 2\pi\int_0^{\sqrt{9-r^2}} (9-x^2-r^2)\,dx \quad \text{(by symmetry)}$$

(continued)

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$= 2\pi \left[(9-r^2)x - \frac{1}{3}x^3 \right]_0^{\sqrt{9-r^2}} = 2\pi \left[(9-r^2)^{3/2} - \frac{1}{3}(9-r^2)^{3/2} \right]$

$= 2\pi (9-r^2)^{3/2}\left[1-\frac{1}{3}\right] = \frac{4\pi}{3}(9-r^2)^{3/2}$

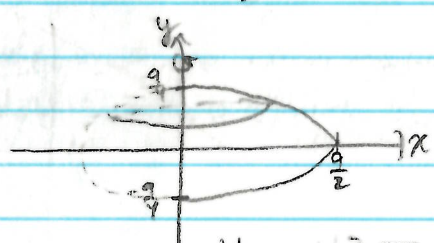
Want: $(0.5)\cdot(36\pi) = \frac{4\pi}{3}(9-r^2)^{3/2}$

$18 = \frac{4}{3}(9-r^2)^{3/2}$

$\frac{27}{2} = (9-r^2)^{3/2} \Rightarrow \left(\frac{27}{2}\right)^{2/3} = 9-r^2$

$\Rightarrow r^2 = 9-\left(\frac{27}{2}\right)^{2/3}$, so $\boxed{r = \sqrt{9-\left(\frac{27}{2}\right)^{2/3}} \approx 1.825 \text{ in}}$

Ex 3. A water tank is in the shape generated by revolving the region enclosed by the right half of the graph $4x^2+16y^2=81$ and the y-axis about the y-axis. Find the depth of water in the tank when it is filled to $\frac{1}{3}$ its volume. (Round to 3 decimals.)



$4x^2+16y^2=81 \Rightarrow 4x^2 = 81-16y^2$

$\Rightarrow x = \pm\sqrt{\frac{81}{4}-4y^2}$

so $r = \sqrt{\frac{81}{4}-4y^2}$

(by symmetry)

$V_{tank} = \pi \int_{-9/4}^{9/4} \left(\sqrt{\frac{81}{4}-4y^2}\right)^2 dy = 2\pi \int_0^{9/4} \left(\frac{81}{4}-4y^2\right)dy$

$= 2\pi \left[\frac{81}{4}y - \frac{4}{3}y^3\right]_0^{9/4} = 2\pi\left[\frac{729}{16} - \frac{243}{16}\right]$

Thus, $\frac{1}{3}V_{tank} = \frac{2}{3}\pi\left(\frac{243}{8}\right)$

Assume the appropriate depth is d. Then disks vary from $y=-\frac{9}{4}$ to $y=-\frac{9}{4}+d$

$V_{\frac{1}{3}} = \pi \int_{-9/4}^{-9/4+d} \left(\sqrt{\frac{81}{4}-4y^2}\right)^2 dy = \pi\left[\frac{81}{4}y - \frac{4}{3}y^3\right]_{-9/4}^{-9/4+d}$

$= \pi\left[\frac{81}{4}\left(-\frac{9}{4}+d\right) - \frac{4}{3}\left(-\frac{9}{4}+d\right)^3 + \frac{729}{16} - \frac{243}{16}\right]$

Want $\pi\left[\frac{81}{4}\left(-\frac{9}{4}+d\right) - \frac{4}{3}\left(-\frac{9}{4}+d\right)^3 + \frac{243}{8}\right] = \frac{2}{3}\pi\left(\frac{243}{8}\right)$

Use Wolfram Alpha to get

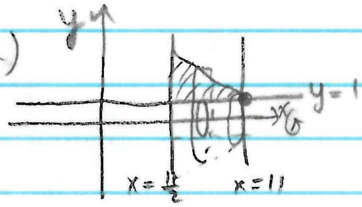
$d \approx -1.368, 1.741, 6.377$; -1.368 and 6.377 are impossible depths

so $\boxed{d \approx 1.741}$

Ex 4. Let S be the region bounded above $x^3y = 1331$, below $y = 1$, on the left by $x = \frac{11}{2}$, on the right by $x = 11$. Find the volume obtained by rotating S about

(a) x-axis (b) $y = 1$ (c) y-axis (d) $x = \frac{11}{2}$

(a)



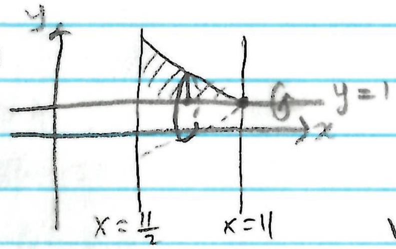
$R = 1331x^{-3}$, $r = 1$

washers vary from $x = \frac{11}{2}$ to $x = 11$

$$V = \pi \int_{\frac{11}{2}}^{11} \left[(1331x^{-3})^2 - (1)^2\right] dx$$

≈ 196.978

(b)



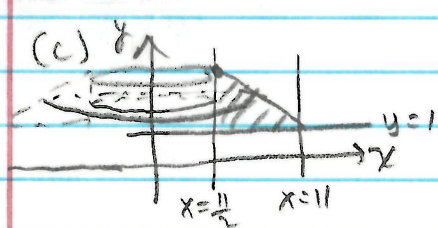
$R = 1331x^{-3} - 1$

disks vary from $x = \frac{11}{2}$ to $x = 1$

$$V = \pi \int_{\frac{11}{2}}^{11} (1331x^{-3} - 1)^2 \, dx$$

(FOIL) ≈ 127.863

(c)



$R = \sqrt[3]{\frac{1331}{y}}$, $r = \frac{11}{2}$

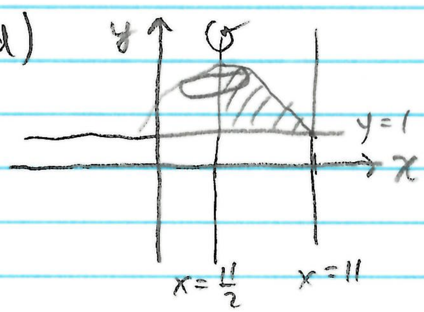
washers vary from $y = 1$ to $y = 8$

(since $(\frac{11}{2})^3 y = 1331 \Rightarrow y = 8$)

$$V = \pi \int_1^8 \left[1331^{2/3} y^{-2/3} - (\tfrac{11}{2})^2\right] dy$$

≈ 475.166

(d)



$R = \sqrt[3]{\frac{1331}{y}} - \frac{11}{2}$

disks vary from $y = 1$ to $y = 8$

$$V = \pi \int_1^8 \left(11y^{-1/3} - \tfrac{11}{2}\right)^2 dy$$

(FOIL)

≈ 95.033