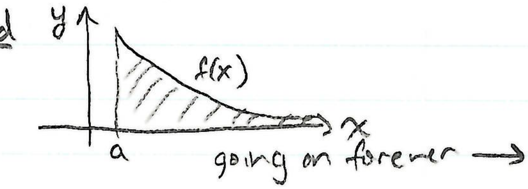
MA 16020

Lesson 15

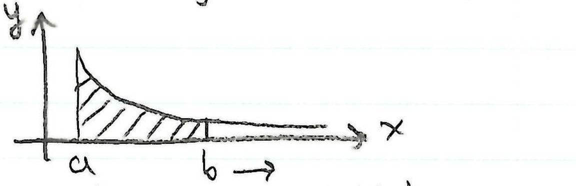
Improper Integrals

Sometimes, it is useful to consider the area under a curve from some point onwards. (This is often useful in probability, diff eqs, etc.)

We consider integrals of the form $\int_a^\infty f(x)\,dx$, which are <u>improper integrals with an infinite bound</u>



By definition, $\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$, if the limit exists. If the limit does not exist (including infinite-value), we say the integral <u>diverges</u>.

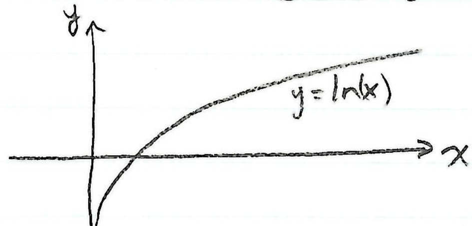


<u>Ex 1</u>. Find the integral, if it converges. $\int_1^\infty \frac{1}{3x-2}\,dx$

$$\lim_{b\to\infty} \int_1^b \frac{1}{3x-2}\,dx = \lim_{b\to\infty} \left[\frac{1}{3}\ln|3x-2|\right]_1^b$$

$$= \lim_{b\to\infty} \left(\frac{1}{3}\ln(3b-2) - \frac{1}{3}\ln(1)\right)$$

$$= \lim_{b\to\infty} \frac{1}{3}\ln(3b-2) - 0$$



as $x \to \infty$, $\ln(x) \to \infty$

(can also see it from $\ln(e^{\text{big }\#}) = \text{big }\#$)

so $\lim_{b\to\infty} \frac{1}{3}\ln(3b-2) = \infty$

The integral $\boxed{\text{diverges}}$

Ex 2. Find the integral, if it converges $\int_0^\infty \frac{5x}{e^{2x}}\,dx$

$$\lim_{b\to\infty}\int_0^b 5xe^{-2x}\,dx$$

integration by parts

$u = 5x \qquad dv = e^{-2x}\,dx$

$du = 5\,dx \qquad v = -\frac{1}{2}e^{-2x}$

$$uv - \int v\,du$$

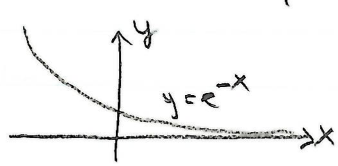
$$\lim_{b\to\infty}\left(-\frac{5}{2}xe^{-2x}\Big|_0^b - \int_0^b -\frac{5}{2}e^{-2x}\,dx\right)$$

$$= \lim_{b\to\infty}\left(-\frac{5}{2}xe^{-2x} - \frac{5}{4}e^{-2x}\right)\Big|_0^b$$

$$= \lim_{b\to\infty}\left(-\frac{5}{2}be^{-2b} - \frac{5}{4}e^{-2b} + 0 - \frac{5}{4}\right)$$

$$= \lim_{b\to\infty} -\frac{5}{2}be^{-2b} + \lim_{b\to\infty} -\frac{5}{4}e^{-2b} + \frac{5}{4}$$

Now, $\lim_{b\to\infty} e^{-b} = 0$



So $\lim_{b\to\infty} e^{-2b} = 0$

But $\lim_{b\to\infty} be^{-2b}$ is of the form $\infty\cdot 0$, which is indeterminate.

Exponential functions "beat" polynomials in limits,

so $= 0$

Thus, $0 + 0 + \frac{5}{4} = \boxed{\frac{5}{4}}$

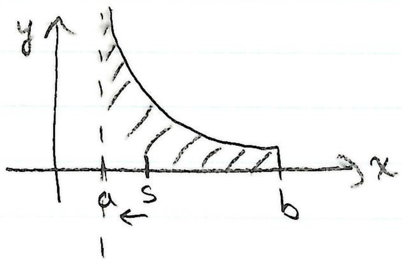
Ex 3 (Skipped for time reasons).
Find the integral, if it converges $\int_1^{\infty} \frac{e^{-2\sqrt{x}}}{2\sqrt{x}}\,dx$

$$\lim_{b\to\infty} \int_1^b \frac{e^{-2\sqrt{x}}}{2\sqrt{x}}\,dx \qquad \text{u-sub} \qquad u = -2\sqrt{x} = -2x^{1/2}$$
$$du = -x^{-1/2}\,dx = -\frac{1}{\sqrt{x}}\,dx$$

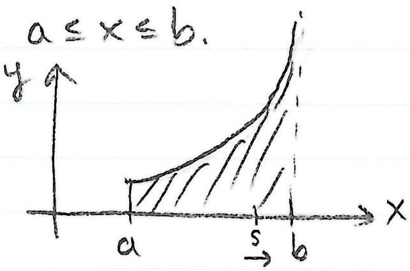
$$\lim_{b\to\infty} \int_{x=1}^{x=b} -\frac{1}{2}e^u\,du = \lim_{b\to\infty} -\frac{1}{2}e^u \Big|_{x=1}^{x=b} = \lim_{b\to\infty} -\frac{1}{2}e^{-2\sqrt{x}}\Big|_1^b$$

$$= \lim_{b\to\infty} -\frac{1}{2}e^{-2\sqrt{b}} + \frac{1}{2}e^{-2} = 0 + \frac{1}{2}e^{-2} = \boxed{\frac{1}{2}e^{-2}} = \boxed{\frac{1}{2e^2}}$$

There are also improper integrals with an infinite discontinuity. Such integrals are of the form $\int_a^b f(x)\,dx$ where $f(x)$ has an infinite discontinuity on $a \le x \le b$.



or



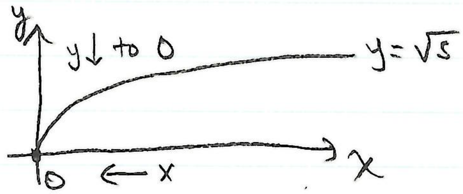
If $f(x)$ has a discontinuity at a, $\int_a^b f(x)\,dx := \lim_{s\to a^+} \int_s^b f(x)\,dx$

If $f(x)$ has a discontinuity at b, $\int_a^b f(x)\,dx := \lim_{s\to b^-} \int_a^s f(x)\,dx$

Ex 4. Find the integral, if it converges $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

$\frac{1}{\sqrt{x}}$ has an infinite discontinuity at 0

$$\lim_{s\to 0^+} \int_s^1 x^{-1/2}\,dx = \lim_{s\to 0^+} (2x^{1/2})\Big|_s^1 = \lim_{s\to 0^+} (2 - 2s^{1/2})$$



$$= 2 - 0$$
$$= \boxed{2}$$

<u>Ex 5.</u> Find the integral, if it converges $\int_0^{\pi} 6\tan\left(\frac{\theta}{2}\right) d\theta$.

$\tan\left(\frac{\theta}{2}\right)$ has infinite discontinuity where $\cos\left(\frac{\theta}{2}\right) = 0$

i.e., when $\frac{\theta}{2} = \frac{\pi}{2} + n\pi$ for integers $n \Leftrightarrow \theta = \pi + 2n\pi$ for integer n.

The only instance in $0 \le x \le \pi$ is π.

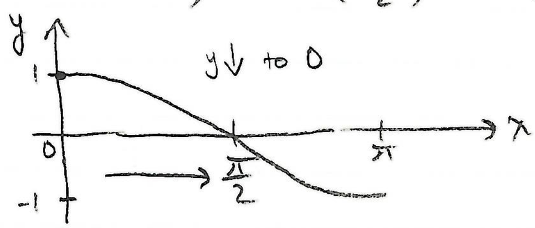
$$\lim_{s\to\pi^-} \int_0^s 6\tan\left(\frac{\theta}{2}\right)d\theta = \lim_{s\to\pi^-}\left[-12\ln\left|\cos\left(\frac{\theta}{2}\right)\right|\right]_0^s \quad \text{(use u-sub)}$$

$$= \lim_{s\to\pi^-}\left[-12\ln\left|\cos\left(\frac{s}{2}\right)\right| + 12\ln\left|\cos(0)\right|\right]$$

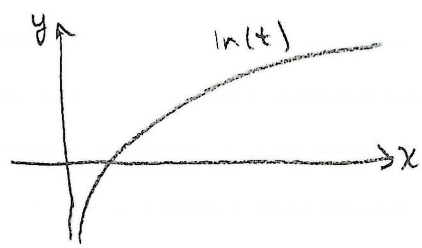
$$= \lim_{s\to\pi^-}\left[-12\ln\left(\cos\left(\frac{s}{2}\right)\right) + 12\underbrace{\ln(1)}_{=0}\right]$$

$$= \lim_{s\to\pi^-} -12\ln\left(\cos\left(\frac{s}{2}\right)\right)$$

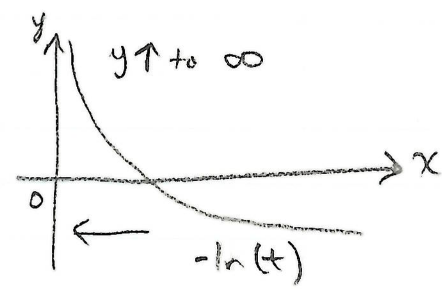
As $s\to\pi^-$, $\cos\left(\frac{s}{2}\right) \to 0^+$



so $= \lim_{t\to 0^+} -12\ln(t)$



So



$= \infty$

so $\boxed{\text{diverges}}$