MA 16020
Lesson 16
Geometric Series (Part 1)

Consider the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^n}+\cdots$.
Initially, you might think that this series has no sum since there are infinitely many terms, but let's see what happens when we add terms together:

$\frac{1}{2}$

$\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{7}{8}$

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=\frac{15}{16}$

etc.

Notice: as we keep adding terms, the partial sums get closer and closer to 1. In fact, adding together the first n terms gives us $\frac{2^n-1}{2^n}=1-\left(\frac{1}{2}\right)^n$ when we add all infinitely many terms together, we should get something like $1-\left(\frac{1}{2}\right)^\infty$. This does not technically make sense, so we say that the sum is $\lim_{n\to\infty} 1-\left(\frac{1}{2}\right)^n=1-0=1$.
Thus, $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=1$.

In the above process, we took partial sums of the series. The nth partial sum of an infinite series is the sum of the first n terms of the series.
If the limit of the partial sums exists, then we say that the series converges and its sum is the limit of the partial sums.
If the limit of the partial sums diverges, then we say that the series diverges.

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Ex 1. Calculate the fourth partial sum of
$$\sum_{n=0}^{\infty} (-1)^n \frac{2^n}{n!}$$

we start with $n=0$, and get the first term is

1st term: $\frac{(-1)^0 2^0}{0!} = \frac{(1)(1)}{(1)} = 1$ $(0! = 1)$

2nd term: $\frac{(-1)^1 2^1}{1!} = \frac{(-1)(2)}{1} = -2$ $(1! = 1)$

3rd term: $\frac{(-1)^2 2^2}{2!} = \frac{(1)(4)}{2} = 2$ $(2! = 2 \cdot 1 = 2)$

4th term: $\frac{(-1)^3 2^3}{3!} = \frac{(-1)(8)}{6} = -\frac{4}{3}$ $(3! = 3 \cdot 2 \cdot 1 = 6)$

So 4th partial sum is: $1 - 2 + 2 - \frac{4}{3} = \boxed{-\frac{1}{3}}$

Ex 2. Write the series in sigma notation
$$6 - \frac{18}{2} + \frac{54}{6} - \frac{162}{24} + \cdots$$

Notice, the denominators are $1!, 2!, 3!, 4!$, etc.
Also, notice that the terms alternate, so we need $(-1)^{power}$. Since the $n=1$ term is positive, need $(-1)^{n+1}$
The numerator is more difficult - but notice each is positive. Factor out a 2 from each to get numerators: 3, 9, 27, 81; i.e., $3^1, 3^2, 3^3, 3^4$.

Thus, $$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, 2 \cdot 3^n}{n!}}$$

On LON-CAPA, these questions are multiple choice, so you can just write out the first several terms of each choice.

Ex 3. Write $5.\overline{3}$ in sigma notation

$$5.\overline{3} = 5 + 0.3 + 0.03 + 0.003 + \ldots$$
$$= 5 + \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots$$
$$= 5 + \sum_{n=0}^{\infty} \frac{3}{10} \cdot \left(\frac{1}{10}\right)^n$$

Geometric Series

A series is called geometric if every term is a specific constant times the previous term.
i.e., $a + ar + ar^2 + ar^3 + \ldots$
r is called the common ratio of the series.

The nth partial sum of a geometric series is $a \cdot \frac{1-r^{n+1}}{1-r}$. Thus, the sum of a geometric series is $\lim_{n\to\infty} a \cdot \frac{1-r^{n+1}}{1-r}$. If $-1 < r < 1$, $\lim_{n\to\infty} r^{n+1} = 0$.

otherwise, the limit diverges.

Thus, $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \ldots = \frac{a}{1-r}$ if $-1 < r < 1$
and diverges if $r \leq -1$ or $r \geq 1$

Notice, a is the first term of the geometric series and r is the second term divided by a.

Our motivating example is a geometric series with $a = \frac{1}{2}$ and $r = \frac{\left(\frac{1}{4}\right)}{\left(\frac{1}{2}\right)} = \frac{1}{2}$, so

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

(since $-1 < \frac{1}{2} < 1$)

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Ex 4. Find the sum of $\sum_{n=1}^{\infty} \frac{3(-1)^{n+1}}{4^{3n}}$

Notice, $\frac{3(-1)^{n+1}}{4^{3n}} = \frac{(3)(-1)(-1)^n}{(4^3)^n} = -3 \cdot \left(-\frac{1}{4^3}\right)^n$

So the series is geometric with $r = -\frac{1}{4^3} = -\frac{1}{64}$

Notice the first term is $\frac{3(-1)^{1+1}}{4^{3\cdot 1}} = \frac{3}{64} = a$

The second term is $\frac{3(-1)^{2+1}}{4^{3\cdot 2}} = \frac{-3}{4096}$, so $r = \frac{-3}{4096} \cdot \frac{64}{3} = -\frac{1}{64}$

Thus, $\frac{a}{1-r} = \frac{\frac{3}{64}}{1-(-\frac{1}{64})} = \frac{\frac{3}{64}}{\frac{65}{64}} = \frac{3}{64} \cdot \frac{64}{65} = \frac{3}{65}$

So $\sum_{n=1}^{\infty} \frac{3(-1)^{n+1}}{4^{3n}} = \boxed{\frac{3}{65}}$ (since $-1 < -\frac{1}{64} < 1$)

Ex 5. Find the sum of $\sum_{n=0}^{\infty} \left(\frac{6}{5^n} + \frac{3}{2^n}\right)$

$\sum_{n=0}^{\infty} \left(\frac{6}{5^n} + \frac{3}{2^n}\right) = \sum_{n=0}^{\infty} \frac{6}{5^n} + \sum_{n=0}^{\infty} \frac{3}{2^n}$ (both geometric)

$\sum_{n=0}^{\infty} \frac{6}{5^n}$ has $a = \frac{6}{5^0} = 6$, $r = \frac{\frac{6}{5}}{6} = \frac{6}{5} \cdot \frac{1}{6} = \frac{1}{5}$ $(-1 < \frac{1}{5} < 1)$

So $\sum_{n=0}^{\infty} \frac{6}{5^n} = \frac{a}{1-r} = \frac{6}{1-\frac{1}{5}} = \frac{6}{\frac{4}{5}} = 6 \cdot \frac{5}{4} = \frac{15}{2}$

$\sum_{n=0}^{\infty} \frac{3}{2^n}$ has $a = \frac{3}{2^0} = 3$, $r = \frac{\frac{3}{2}}{3} = \frac{3}{2} \cdot \frac{1}{3} = \frac{1}{2}$ $(-1 < \frac{1}{2} < 1)$

So $\sum_{n=0}^{\infty} \frac{3}{2^n} = \frac{a}{1-r} = \frac{3}{1-\frac{1}{2}} = \frac{3}{\frac{1}{2}} = 6$

So $\sum_{n=0}^{\infty} \left(\frac{6}{5^n} + \frac{3}{2^n}\right) = \frac{15}{2} + 6 = \boxed{\frac{27}{2}}$

Warning: Make sure to check $-1 < r < 1$ before using the sum formula. $\sum_{n=0}^{\infty} 2^n = 1+2+4+8+\ldots$ is geometric with $a=1$, $r=2$, so it diverges (to ∞, which is clear) but $\frac{a}{1-r} = \frac{1}{1-2} = \frac{1}{-1} = -1$, which is clearly not right.