MA 16020

Lesson 17

Applications for Geometric Series

For applications, carefully write out the first several terms of the series to identify the pattern. Then find the sum.

Ex 1. A patient is given an injection of 55mg every 24 hours. After t days, the fraction of drug remaining is $f(t) = 2^{-t/3}$. If the injections continue indefinitely, how much will be in the patient's body in the long run just prior to an injection? (Round to 1 decimal place)

In the long run, the patient has received injections for a long time. Just before an injection, the patient has not had an injection from today, so the drug in the body is...

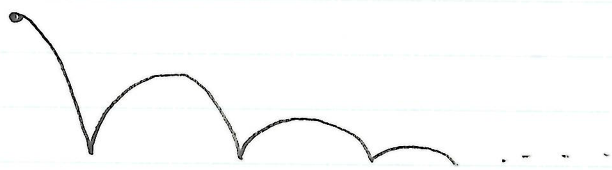
$$\begin{pmatrix}\text{amount remaining}\\ \text{from 1 day ago}\end{pmatrix} + \begin{pmatrix}\text{amount remaining}\\ \text{from 2 days ago}\end{pmatrix} + \begin{pmatrix}\text{amount remaining}\\ \text{from 3 days ago}\end{pmatrix} + \dots$$

$$\underbrace{55}_{\text{injection}} \cdot \underbrace{2^{-1/3}}_{\text{fraction remaining after } t=1 \text{ day}} + \underbrace{55}_{\text{injection}} \cdot \underbrace{2^{-2/3}}_{\text{fraction remaining after } t=2 \text{ days}} + 55 \cdot 2^{-3/3} + \dots$$

$a = 55 \cdot 2^{-1/3}$, $r = 2^{-1/3}$ so $\dfrac{a}{1-r} = \dfrac{55 \cdot 2^{-1/3}}{1-2^{-1/3}} \approx 211.6$ mg

(Similar to Mars Colony Problem)

Ex 2. Every time a ball bounces, it bounces to a height of rh, where h is the height it fell from. Find the total distance the ball travels if $r = 0.7$ and it is dropped from a height of 12m.



$12\downarrow \quad \uparrow 0.7\cdot 12 \quad \downarrow 0.7\cdot 12 \quad \uparrow (0.7)^2\cdot 12 \quad \downarrow (0.7)^2\cdot 12 \quad \uparrow \ \downarrow$

$$12 + 0.7\cdot 12 + 0.7\cdot 12 + 0.7^2\cdot 12 + 0.7^2\cdot 12 + \cdots$$

$$= \underbrace{12}_{\text{doesn't follow pattern}} + \underbrace{0.7\cdot 2\cdot 12 + 0.7^2\cdot 2\cdot 12 + 0.7^3\cdot 2\cdot 12 + \cdots}_{\text{geometric series}}$$

$a = 0.7\cdot 2\cdot 12$, $r = 0.7$

$$12 + \frac{a}{1-r} = 12 + \frac{0.7\cdot 2\cdot 12}{1-0.7} = \boxed{68 \text{ meters}}$$

Ex 3. In a certain country, 55% of all income the people receive is spent and the other 45% is saved. What is the total amount of spending in the long run generated by a stimulus of \$80 billion? (Round to 2 decimals)

Notice that what is spent in a given year is also income in the economy that same year (for the people receiving that spent money), so they will spend 55% of that next year.

$$\binom{\text{original}}{\text{amount}} + \left(\text{spent year}_1\right) + \left(\text{spent year}_2\right) + \left(\text{spent year}_3\right) + \dots$$

$$\$80 \text{ bill} + 0.55 \cdot 80 \text{ bill} + 0.55^2 \cdot 80 \text{ bill} + 0.55^3 \cdot 80 \text{ bill}$$

$$a = 80 \text{ bill}, \quad r = 0.55 \quad \text{so} \quad \frac{a}{1-r} = \frac{80 \text{ bill}}{1-0.55} \approx \boxed{\$177.78 \text{ billion}}$$

Ex 4. How much money should you invest today at an annual interest rate of 8.2% compounded continuously so that, starting 3 years from now, you can make annual withdrawals of \$3100 in perpetuity? (Round to the nearest cent.)

You invest a lump sum now, and want to withdraw \$3100 every year.
Continuous compound interest formula: $A = Pe^{rt}$ where A is amount, P is principal, r is interest rate; so $P = Ae^{-rt}$

Every year, you want to have $A = 3100$; $r = 0.082$

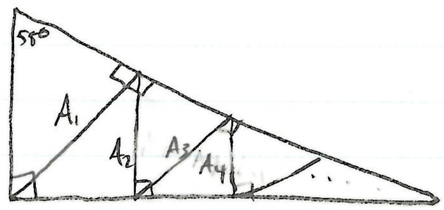
To be able to withdraw 3100 in 3 years, you need to invest $3100e^{-0.082\cdot 3}$ now
4 years, $3100e^{-0.082\cdot 4}$
5 years, $3100e^{-0.082\cdot 5}$
ect.

$$3100e^{-0.082\cdot 3} + 3100e^{-0.082\cdot 4} + 3100e^{-0.082\cdot 5} + \dots$$

$a = 3100e^{-0.082\cdot 3}$
$r = e^{-0.082}$

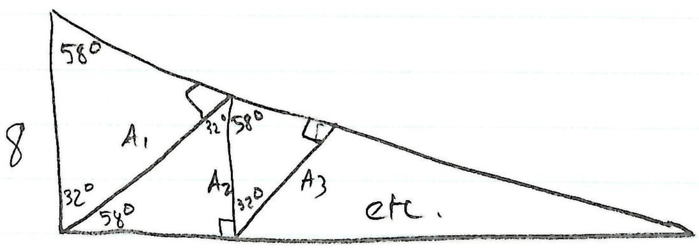
$$\frac{3100e^{-0.082\cdot 3}}{1-e^{-0.082}} \approx \boxed{\$30{,}789.02}$$

Ex 5. In a right triangle, a series of perpendicular line segments are drawn starting with an altitude using the vertex of the right angle, Then consequently drawing altitudes in the same way - from the new right angle in the triangle containing the smaller angle. Find the sum of the lengths of all of these perpendicular line segments if one angle is $58°$ and the side adjacent to this angle is 8 meters long.

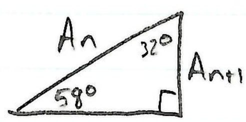


Want $A_1 + A_2 + A_3 + A_4 + \ldots$

We can find angles for each of these triangles



If we let $8 = A_0$, notice all little triangles look like



(for $n \geq 0$)

So $\sin(58°) = \dfrac{A_{n+1}}{A_n}$ giving $A_{n+1} = A_n \cdot \sin(58°)$

$A_0 = 8$

$A_1 = A_0 \cdot \sin(55°) = 8 \cdot \sin(55°)$

$A_2 = A_1 \cdot \sin(55°) = 8 \cdot (\sin(55°))^2$

$A_3 = A_2 \cdot \sin(55°) = 8 \cdot (\sin(55°))^3$

So for $A_1 + A_2 + A_3 + A_4 + \ldots$

$a = 8 \cdot \sin(55°)$ and $r = \sin(55°)$

$$\frac{a}{1-r} = \frac{8 \cdot \sin(55°)}{1 - \sin(55°)} \approx \boxed{44.65 \text{ m}}$$