MA 16020
Lesson 19
Partial Derivatives

For single variable functions, there is only one direction in which you can change your domain (x-direction). For multivariate functions, you can move in any direction on the plane, but in particular, you can move in the x-direction and the y-direction.

For single variable functions $y = f(x)$, $\frac{dy}{dx}$ tells how much y changes as you move in the x-direction.

For multivariate functions $z = f(x,y)$ we have <u>partial derivatives</u> $\frac{\partial z}{\partial x} = f_x(x,y)$ telling how much z changes as you move in the x-direction and $\frac{\partial z}{\partial y} = f_y(x,y)$ telling how much z changes as you move in the y-direction.

To compute a partial derivative with respect to a certain variable, treat all other variables as if they are constants and differentiate with respect to your certain variable.

e.g., for $z = f(x,y)$,

$\frac{\partial z}{\partial x} = f_x(x,y)$: treat y as a constant
differentiate with respect to x

$\frac{\partial z}{\partial y} = f_y(x,y)$: treat x as a constant
differentiate with respect to y

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Ex 1. Compute f_x and f_y for $f(x,y) = x^2 + 3xy + y^2$

f_x : treat y as a constant

$$f = x^2 + \underbrace{3y}_{\text{constant}}x + \underbrace{y^2}_{\text{constant}} \longrightarrow f_x = 2x + 3y + 0 = 2x + 3y$$

f_y : treat x as a constant

$$f = \underbrace{x^2}_{\text{constant}} + \underbrace{3x}_{\text{constant}}y + y^2 \longrightarrow f_y = 0 + 3x + 2y = 3x + 2y$$

$$\boxed{f_x(x,y) = 2x + 3y} \qquad \boxed{f_y(x,y) = 3x + 2y}$$

Ex 2. Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $z = e^{(x+y)(xy)}$

$\frac{\partial z}{\partial x}$: treat y as a constant

By Chain Rule: $\frac{\partial z}{\partial x} = e^{(x+y)(xy)} \cdot \frac{\partial}{\partial x}\left((x+y)(xy)\right)$

By Product Rule: $e^{(x+y)(xy)}\left[(x+y)\frac{\partial}{\partial x}(x\underset{\sqcup}{y}) + \left[\frac{\partial}{\partial x}(x+\underset{\sqcup}{y})\right](xy)\right]$

$$= e^{(x+y)(xy)}\left[(x+y)(y) + (1+0)(xy)\right]$$

$$\frac{\partial z}{\partial x} = e^{(x+y)(xy)} \cdot (xy + y^2 + xy) = \boxed{e^{(x+y)(xy)} \cdot (y^2 + 2xy)}$$

$\frac{\partial z}{\partial y}$: treat x as a constant

By Chain Rule: $\frac{\partial z}{\partial y} = e^{(x+y)(xy)} \cdot \frac{\partial}{\partial y}\left((x+y)(xy)\right)$

By product Rule: $e^{(x+y)(xy)}\left[(x+y)\cdot\frac{\partial}{\partial y}(\underset{\sqcup}{x}y) + \left[\frac{\partial}{\partial y}(\underset{\sqcup}{x}+y)\right](xy)\right]$

$$= e^{(x+y)(xy)}\left[(x+y)(x) + (0+1)(xy)\right]$$

$$\frac{\partial z}{\partial y} = e^{(x+y)(xy)} \cdot (x^2 + xy + xy) = \boxed{e^{(x+y)(xy)} \cdot (x^2 + 2xy)}$$

Ex 3. Find f_x and f_y for $f(x,y) = \ln(x^2 + xy)$

f_x : y as constant. Chain rule: $f_x = \dfrac{1}{x^2+xy} \cdot \dfrac{\partial}{\partial x}(x^2 + xy)$

$$= \frac{1}{x^2+y^2} \cdot (2x+y) = \boxed{\frac{2x+y}{x^2+y^2}}$$

f_y : x as conltant. Chain rule: $f_y = \dfrac{1}{x^2+xy} \cdot \dfrac{\partial}{\partial y}(x^2 + xy)$

$$= \frac{1}{x^2+xy} \cdot (0+x) = \frac{x}{x^2+xy} = \boxed{\frac{1}{x+y}}$$

Ex 4. Find $f_x(1,2)$ and $f_y(1,2)$ for

$$f(x,y) = \frac{xy}{x+y+3}$$

f_x : y constant, Quotient Rule: $f_x = \dfrac{(x+y+3)\frac{\partial}{\partial x}(xy) - (xy)\frac{\partial}{\partial x}(x+y+3)}{(x+y+3)^2}$

$$= \frac{(x+y+3)(y) - (xy)(1+0+0)}{(x+y+3)^2} = \frac{xy + y^2 + 3y - xy}{(x+y+3)^2}$$

$$f_x = \frac{y^2+3y}{(x+y+3)^2}, \quad f_x(1,2) = \frac{(2)^2 + 3(2)}{((1)+(2)+3)^2} = \frac{4+6}{6^2} = \frac{10}{36} = \boxed{\frac{5}{18}}$$

f_y : x constant Quotient rule: $f_y = \dfrac{(x+y+3)\frac{\partial}{\partial y}(xy) - (xy)\frac{\partial}{\partial y}(x+y+3)}{(x+y+3)^2}$

$$= \frac{(x+y+3)(x) - (xy)(0+1+0)}{(x+y+3)^2} = \frac{x^2+xy+3x-xy}{(x+y+3)^2}$$

$$f_y = \frac{x^2+3x}{(x+y+3)^2}, \quad f_y(1,2) = \frac{(1)^2+3(1)}{((1)+(2)+3)^2} = \frac{1+3}{6^2} = \frac{4}{36} = \boxed{\frac{1}{9}}$$

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Ex 5. Find $\frac{\partial R}{\partial M}$ and $\frac{\partial R}{\partial C}$ for
$$R(M, C) = \frac{M}{C} \cdot 100$$

$\frac{\partial R}{\partial M}$: C constant : $R = \underbrace{\frac{100}{C}} \cdot M$, $\frac{\partial R}{\partial M} = \boxed{\frac{100}{C}}$

$\frac{\partial R}{\partial C}$: M constant : $R = \underbrace{100M} \cdot C^{-1}$, $\frac{\partial R}{\partial C} = -100M \cdot C^{-2}$
$= \boxed{\frac{-100M}{C^2}}$