MA 16020

Lesson 2

Substitution (Part 2)

When doing a definite integral $\int_a^b f(x)\,dx$, notice that a and b are x-values. As such, when we do a u-substitution, we should change these to "u-values." This can be done by plugging a and b into the "u = inner function" equation.

Ex 1. $\int_1^3 4(x^2-2)(x^3-6x)^2\,dx$

Most inside function is x^3-6x

$u = x^3-6x, \quad du = (3x^2-6)\,dx$
$\qquad\qquad\qquad\quad = 3(x^2-2)\,dx$

so $\frac{1}{3}du = (x^2-2)\,dx$

$\frac{4}{3}\int_a^b u^2\,du$

For a, plug in x = 1 into u: $a = (1)^3-6(1) = 1-6 = -5$

For b, plug in x = 3 into u: $b = (3)^3-6(3) = 27-18 = 9$

$\frac{4}{3}\int_{-5}^{9} u^2\,du$

$= \frac{4}{3}\left[\frac{1}{3}u^3\right]\Big|_{-5}^{9} = \frac{4}{3}\left[\frac{1}{3}(9)^3 - \frac{1}{3}(-5)^3\right]$

$= \frac{4}{3}\left[243 + \frac{125}{3}\right]$

$= \boxed{\frac{3416}{9}}$

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Substitution (part 2)

Sometimes, after we substitute u and du, we may have some x's left over. When this happens, we may be able to continue with substitution by taking our "u = inner function" equation and performing algebraic manipulations on it until we have something which matches the left over x's. We can then use that to substitute.

Ex 2. $\int x^2\sqrt{3x-3}\,dx$

$u = 3x-3, \quad du = 3\,dx \quad \text{so} \quad \frac{1}{3}du = dx$

$\frac{1}{3}\int x^2\sqrt{u}\,du$

Need to get x^2, $\quad u = 3x-3$, so $\quad u+3 = 3x$

so $\frac{1}{3}u+1 = x$

So $(\frac{1}{3}u+1)^2 = x^2$

$(\frac{1}{3}u+1)^2 = \frac{1}{9}u^2 + \frac{2}{3}u + 1$

$$\frac{1}{3}\int\left(\frac{1}{9}u^2 + \frac{2}{3}u + 1\right)u^{1/2}\,du$$

$$= \frac{1}{3}\int\left(\frac{1}{9}u^{5/2} + \frac{2}{3}u^{3/2} + u^{1/2}\right)du$$

$$= \frac{1}{3}\left[\frac{1}{9}\cdot\frac{2}{7}u^{7/2} + \frac{2}{3}\cdot\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right] + C$$

(distribute $\frac{1}{3}$)

$$= \frac{2}{189}u^{7/2} + \frac{4}{45}u^{5/2} + \frac{2}{9}u^{3/2} + C$$

$$= \boxed{\frac{2}{189}(3x-3)^{7/2} + \frac{4}{45}(3x-3)^{5/2} + \frac{2}{9}(3x-3)^{3/2} + C}$$

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Ex 3. $\int \frac{3x}{\sqrt{x+2}}\,dx$

$u = x+2,\quad du = dx$

$\int \frac{3x}{\sqrt{u}}\,du$

$u = x+2$, so $u-2 = x$, so $3u-6 = 3x$

$$\int \frac{3u-6}{\sqrt{u}}\,du = \int (3u-6)u^{-1/2}\,du$$
$$= \int (3u^{1/2} - 6u^{-1/2})\,du$$
$$= 3\cdot\frac{2}{3}u^{3/2} - 6\cdot 2u^{1/2} + C$$
$$= 2u^{3/2} - 12u^{1/2} + C$$
$$= 2(x+2)^{3/2} - 12(x+2)^{1/2} + C$$

Change of Value and Average Value

Given a function $F(x)$, the change in $F(x)$ from a to b is $F(b) - F(a) = \int_a^b F'(x)\,dx$.
Thus, if you want to know the actual change of a function $f(x)$ and you know its derivative $f'(x)$, then the change is $\int_a^b f'(x)\,dx$

Given a function $f(x)$, the average value of $f(x)$ for $a \le x \le b$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$. Thus, if you want to know the average value of a function $f(x)$, take the actual function itself in the integral.

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<u>Ex 4</u>. A cup of coffee is 190° F, and its temperature changes at a rate of $T'(t) = -3e^{-0.03t}$ °F/min.

(a) How much does the temperature change in the first 5 minutes? (Round to 2 decimal places.)

$$\int_0^5 -3e^{-0.03t}\,dt$$

$$u = -0.03t,\quad du = -0.03\,dt,\quad \frac{1}{-0.03}du = dt$$

For a, $a = -0.03(0) = 0$,

For b, $b = -0.03(5) = -0.15$

$$100\int_0^{-0.15} e^u\,du = 100\left[e^{-0.15} - e^0\right] \approx \boxed{-13.93\text{ °F}}$$

(b) What is the average temperature during the first 5 minutes? (Round to 2 decimal places)

Need $\frac{1}{5-0}\int_0^5 T(t)\,dt$, but we don't have $T(t)$.

$$T(t) = \int -3e^{-0.03t}\,dt$$

$$u = -0.03t,\quad du = -0.03\,dt,\quad \frac{1}{-0.03}du = dt$$

$$100\int e^u\,du = 100e^{-0.03t} + C$$

$$190 = 100e^{-0.03(0)} + C \Rightarrow 190 = 100 + C \Rightarrow C = 90$$

$$T(t) = 100e^{-0.03t} + 90$$

So Ave $= \frac{1}{5}\int_0^5 (100e^{-0.03t} + 90)\,dt$

$$= \underbrace{\frac{1}{5}\int_0^5 100e^{-0.03t}\,dt} + \frac{1}{5}\int_0^5 90\,dt$$

$$u = -0.03t,\quad du = -0.03\,dt$$

$$= \frac{1}{5}\int_0^{-0.15} -\frac{100}{0.03}e^u\,du + \frac{1}{5}\int_0^5 90\,dt$$

$$-\frac{20}{0.03}\left[e^u\right]_0^{-0.15} + \frac{1}{5}\left[90t\right]_0^5$$

$$-\frac{20}{0.03}\left[e^{-0.15} - e^0\right] + \frac{1}{5}\left[90(5) - 90(0)\right]$$

$$\approx \boxed{182.86\text{ °F}}$$