MA 16020
Lesson 20
Higher Order Partial Derivatives

Recall: In calc 1, you could take the 2nd derivative of a function $f(x)$ by taking the derivative of the derivative.
Same process here: We get "four" 2nd order partial derivatives:
$f_{xx} = (f_x)_x$, $f_{yy} = (f_y)_y$, $f_{xy} = (f_x)_y$, $f_{yx} = (f_y)_x$

Clairaut's Theorem. For nice enough functions (everything we deal with in this class) $f_{xy} = f_{yx}$.

Sometimes f_{xy} or f_{yx} is easier to compute, so choose the easier method.

Ex 1. Find f_{xx}, f_{yy}, f_{xy} of $f(x,y) = 7x\cos(7y)$

$f_x = 7\cos(7y)$, $f_y = 7x(-\sin(7y)\cdot 7) = -49x\sin(7y)$

$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}(7\cos(7y)) = \boxed{0}$

$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}(-49x\sin(7y)) = -49x(\cos(7y)\cdot 7) = \boxed{-343x\cos(7y)}$

$f_{xy} = (f_x)_y$ or $(f_y)_x$ $\quad (f_x)_y = \frac{\partial}{\partial y}(7\cos(7y)) = -7\sin(7y)\cdot 7 = -49\sin(7y)$

or $(f_y)_x = \frac{\partial}{\partial x}(-49x\sin(7y)) = \boxed{-49\sin(7y)}$

Ex 2. Find $f_{xy}(-1,3)$ for $f(x,y) = \frac{5x\ln y}{y^2-2}$. (Round 2 decimals)

$f_x = \frac{5\ln y}{y^2-2}$, $f_{xy} = \frac{(y^2-2)(\frac{5}{y}) - (5\ln y)(2y)}{(y^2-2)^2}$

$$= \frac{5y - \frac{10}{y} - 10y\ln y}{(y^2-2)^2}$$

$f_{xy}(-1,3) = \frac{5(3) - \frac{10}{3} - 10(3)\ln(3)}{((3)^2-2)^2} \approx \boxed{-0.43}$

Ex 3. Find f_{xx}, f_{yy}, and f_{xy} of $f(x,y) = y e^{\sin(3x)}$

$$f_x = y e^{\sin(3x)} \cdot \cos(3x) \cdot 3 = 3y\cos(3x)e^{\sin(3x)}$$
$$f_y = e^{\sin(3x)}$$

$$f_{xx} = (f_x)_x = (3y\cos(3x))(e^{\sin(3x)} \cdot \cos(3x) \cdot 3) + (-3y\sin(3x)\cdot 3)(e^{\sin(3x)})$$
$$= 9y\cos^2(3x)e^{\sin(3x)} - 9y\sin(3x)e^{\sin(3x)}$$

$$f_{yy} = (f_y)_y = 0$$

$$f_{xy} = (f_x)_y = 3\cos(3x)e^{\sin(3x)}$$

Ex 4. Find f_{xx}, f_{yy}, f_{xy} for $f(x,y) = -\frac{x\ln(2xy)}{3y}$

$$f_x = -\frac{1}{3y} \cdot \frac{\partial}{\partial x}(x\ln(2xy)) = -\frac{1}{3y}\left[x \cdot \frac{1}{\cancel{2xy}} \cdot \cancel{2y} + \ln(2xy)\right]$$
$$= \frac{-1-\ln(2xy)}{3y}$$

$$f_y = \frac{(3y)\left(\frac{-x}{2xy} \cdot 2x\right) - (-x\ln(2xy))(3)}{(3y)^2} = \frac{-3x + 3x\ln(2xy)}{9y^2}$$
3

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(-\frac{1}{3y} - \frac{1}{3y} \cdot \ln(2xy)\right) = 0 - \frac{1}{3y} \cdot \frac{1}{\cancel{2}xy} \cdot \cancel{2y} = \boxed{-\frac{1}{3xy}}$$

$$f_{yy} = \frac{(3y^2)\left(0 + \frac{x}{2xy} \cdot 2x\right) - (-x + x\ln(2xy))(6y)}{(3y^2)^2} = \frac{3xy + 6xy - 6xy\ln(2xy)}{9y^4}$$
$$= \frac{9xy - 6xy\ln(2xy)}{9y^4} = \boxed{\frac{3x - 2x\ln(2xy)}{3y^3}}$$

$$f_{xy} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{-x + x\ln(2xy)}{3y^2}\right) = \frac{1}{3y^2}\left[-1 + x \cdot \frac{1}{\cancel{2xy}} \cdot \cancel{2y} + \ln(2xy)\right]$$
$$= \frac{1}{3y^2}\left[-1 + 1 + \ln(2xy)\right]$$
$$= \boxed{\frac{\ln(2xy)}{3y^2}}$$

MA 16020
Lesson 20

Ex 5. If $f_{xx}(0,1) = 12$ for $f(x,y) = 5ye^{ax^2}$, find the constant a.

$f_x = 5ye^{ax^2} \cdot 2ax = 10axye^{ax^2}$

$f_{xx} = (10axy)(e^{ax^2} \cdot 2ax) + (e^{ax^2})(10ay)$
$= 20a^2x^2ye^{ax^2} + 10aye^{ax^2}$

$12 = f_{xx}(0,1) = 20a^2(0)^2(1)e^0 + 10a(1)e^0$
$= 0 + 10a$
$12 = 10a, \quad a = \frac{12}{10} = \boxed{\frac{6}{5}}$