Recall, for \( y = f(x) \), \( \frac{dy}{dx} = f'(x) \), so \( dy = f'(x) \, dx \). Therefore, you can approximate a small change \( \Delta y \approx f'(x) \, \Delta x \).

For \( z = f(x, y) \), we get partial differentials:
- \( \frac{\partial z}{\partial x} \) (with respect to \( x \)) = \( \frac{\partial z}{\partial y} \) (with respect to \( y \)) = \( \frac{\partial z}{\partial y} \) dy

Thus, we get the total differential:
\[
\Delta z = \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y
\]

Thus, you can approximate changes in \( z \) given small changes in \( y \) and \( x \):
\[
\Delta z \approx \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y
\]

at the value \((x_{\text{initial}}, y_{\text{initial}})\)

\( \Delta \text{variable} = \text{change in variable} = (\text{variable final value}) - (\text{variable initial value}) \)

**Ex 1.** Estimate \( \sqrt{2.2^2 + 3.1} - \sqrt{2^2 + 3} \) using differentials.

(Round to 2 decimal places)

Let \( z = f(x, y) = \sqrt{x^2 + y^2} \)

Then \( \Delta z = \sqrt{2.2^2 + 3.1} - \sqrt{2^2 + 3} \)

\( \Delta x = 2.2 - 2 = 0.2 \), \( \Delta y = 3.1 - 3 = 0.1 \)

\( \frac{\partial z}{\partial x} = \frac{1}{2} (x^2 + y^2)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2 + y^2}} \) when \( x = 2, y = 3 \):

\( \frac{\partial z}{\partial x} = \frac{2}{\sqrt{7}} \)

\( \frac{\partial z}{\partial y} = \frac{1}{2} (x^2 + y^2)^{-1/2} = \frac{1}{2\sqrt{x^2 + y^2}} \) when \( x = 2, y = 3 \):

\( \frac{\partial z}{\partial y} = \frac{1}{2\sqrt{7}} \)

\( \Delta z \approx \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y = \frac{2}{\sqrt{7}} \cdot 0.2 + \frac{1}{2\sqrt{7}} \cdot 0.1 \)

\( \approx 0.17 \)
Ex 2. An ideal gas satisfies $PV = 0.5T$, where $P$ is pressure, $V$ is volume, $T$ is temperature. A scientist measures volume as $2 \text{ m}^3$ with an error of $0.1 \text{ m}^3$ and temperature as 200 Kelvin with an error of 5 Kelvin. What is the maximum error in the estimated pressure? (Round to 2 decimals)

$$P = 0.5TV^{-1} \quad \text{so} \quad \frac{dP}{dT} = 0.5V^{-1} = \frac{0.5}{V}, \quad \frac{dP}{dV} = -0.5TV^{-2} = -\frac{0.5T}{V^2}$$

$T_{\text{measured}} = 200, \quad V_{\text{measured}} = 2, \quad \Delta T = \pm 5, \quad \Delta V = \pm 0.1$

$$\Delta P \approx \frac{0.5}{V} \Delta T + \frac{-0.5T}{V^2} \Delta V = \frac{0.5}{2} (\pm 5) + \frac{-0.5(200)}{2^2} (\pm 0.1)$$

four possibilities: $-1.25, -3.75, 1.25, \text{or} 3.75$

maximal error is when you get $3.75$ (or $-3.75$) kPa

Ex 3. A soup can with height $h$ cm and radius $r$ cm has volume $V = \pi r^2 h$. Currently, it has $h = 8$ cm and $r = 3$ cm. If they want to decrease the height by 0.1 cm and keep the volume the same, use differentials to estimate how much they should change the radius. (Round to 3 decimals)

$$\frac{dV}{dr} = 2\pi rh, \quad \frac{dV}{dh} = \pi r^2, \quad r_{\text{initial}} = 3, \quad h_{\text{initial}} = 8, \quad \Delta h = -0.1, \quad \Delta V = 0\quad \text{(decrease)}$$

Find $\Delta r$.

$$\Delta V \approx 2\pi rh \cdot \Delta r + \pi r^2 \cdot \Delta h$$

$$0 \approx 2\pi(3)(8) \cdot \Delta r + \pi (3)^2 (-0.1)$$

$$0 \approx 48\pi \Delta r - 0.9\pi$$

$$\Delta r \approx \frac{0.9\pi}{48\pi} \approx 0.019$$

Since $\Delta r$ is positive, increase radius by about 0.019 cm.
Ex 4. A company has \( P(x, y) = 20x^{3/4}y^{1/4} \) thousand units produced where \( x \) is number of employees and \( y \) is expenditures in thousands of dollars. Suppose they wish to reduce the employees from 100 to 90 and increase expenditures from $15,000 to $18,000.

(a) Estimate the change in productivity due to change in employees.

Want \( \Delta P \) wrt \( x \) \( \approx \frac{2}{3} \frac{P}{dx} \Delta x \), \( \Delta x = 90 - 100 = -10 \)
\( \frac{2}{3} \frac{P}{dx} = 15x^{-1/4}y^{-1/4} \) so \( \Delta P \) wrt \( x \approx 15 \cdot (100)^{-1/4} \cdot (15)^{-1/4} \cdot (-10) \)
\( \approx -93.350 \) thousand units

So a decrease of 93.350 thousand units

(b) Estimate the change in productivity due to change in expenditures.

Want \( \Delta P \) wrt \( y \) \( \approx \frac{2}{3} \frac{P}{dy} \Delta y \), \( \Delta y = 18 - 15 = 3 \)
\( \frac{2}{3} \frac{P}{dy} = 5x^{3/4}y^{-3/4} \) so \( \Delta P \) wrt \( y \approx 5 \cdot (100)^{3/4} \cdot (15)^{-3/4} \cdot (3) \)
\( \approx 62.233 \) thousand units

So an increase of 62.233 thousand units

(c) Estimate the total change in productivity.

\( \Delta P = \Delta P \) wrt \( x \) + \( \Delta P \) wrt \( y \approx -93.350 + 62.233 \approx -31.117 \)
so a decrease of 31.117 thousand units

Ex 5. A can of height \( h \) cm and radius \( r \) cm is in production. The materials for the can cost 0.002 cents per cm\(^2\) and the liquid inside costs 0.001 cents per cm\(^3\). Estimate the change in cost by increasing height by 0.2 cm and decreasing radius by 0.3 cm if \( r = 4 \) cm and \( h = 9 \) cm.

\[
SA = 2\pi r^2 + 2\pi rh, \quad V = \pi r^2 h
\]
so \( C = 0.002(2\pi r^2 + 2\pi rh) + 0.001(\pi r^2 h) \)
\( \frac{2C}{dr} = 0.002(4\pi r + 2\pi h) + 0.001(2\pi rh) \)
\( \frac{2C}{dh} = 0.002(2\pi r) \)
\( AC \approx [0.002(2\pi r + 2\pi h) + 0.001(\pi r^2)] \Delta r + [0.002(2\pi r) + 0.001(\pi r^2)] \Delta h \)
\( \approx [0.002(2\pi(4) + 2\pi(9)) + 0.001(\pi(4)^2(9))](-0.3) + [0.002(2\pi(4)) + 0.001(\pi(4)^2)](0.2) \)
\( \approx -0.112, \) so approx 0.112 cent decrease
Relative percent error of $f$ at $c$ is

$$\left( \frac{\text{max error of } f}{f(c)} \right) \times 100 \%$$

**Ex 6.** If $S = \frac{A}{A-W}$, $A$ is measured to be 2.8 with max error of 0.01, $W$ is measured to be 2.2 with max error of 0.05, approximate the relative percentage error of $S$.

$$\frac{\partial S}{\partial A} = \frac{(A-W)(1)-(A)(1)}{(A-W)^2} = -\frac{W}{(A-W)^2}$$

$$\frac{\partial S}{\partial W} = \frac{\partial}{\partial W} (A(A-W)^{-1}) = -A(A-W)^{-2}(-1) = \frac{A}{(A-W)^2}$$

So

$$\Delta S \approx -\frac{W}{(A-W)^2} \Delta A + \frac{A}{(A-W)^2} \Delta W$$

error of $S \approx \frac{-2.2}{(2.8-2.2)^2} (\pm 0.01) + \frac{2.8}{(2.8-2.2)^2} (\pm 0.05)$

$$\approx \pm 0.328 \text{ or } \pm 0.45$$

max error is 0.45

Calculate $S = \frac{A}{A-W} = \frac{2.8}{2.8-2.2} = \frac{14}{3}$

relative percent error = $\left( \frac{0.45}{(\frac{14}{3})} \right) \times 100 \% \approx 9.64 \%$