MA 16020
Lesson 21
Multivariate Differentials

Recall, for $y = f(x)$, $\frac{dy}{dx} = f'(x)$, so $dy = f'(x)\,dx$.
Therefore, you can approximate a small change $\Delta y \approx f'(x)\,\Delta x$.

For $z = f(x,y)$, we get partial differentials

∂z (with respect to x) $= \frac{\partial z}{\partial x}\,dx$

∂z (with respect to y) $= \frac{\partial z}{\partial y}\,dy$

Thus, we get the total differential

$$dz = \partial z\,(\text{wrt } x) + \partial z\,(\text{wrt } y) = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

Thus, you can approximate changes in z given small changes in y and x:

$$\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y \quad \text{at the value } (x_{\text{initial}}, y_{\text{initial}})$$

Δvariable = change in variable

= (variable final value) − (variable initial value)

Ex 1. Estimate $\sqrt{2.2^2+3.1} - \sqrt{2^2+3}$ using differentials.
(Round to 2 decimal places)

Let $z = f(x,y) = \sqrt{x^2+y}$

Then $\Delta z = \sqrt{2.2^2+3.1} - \sqrt{2^2+3}$

$\Delta x = 2.2 - 2 = 0.2$, $\Delta y = 3.1 - 3 = 0.1$

$\frac{\partial z}{\partial x} = \frac{1}{2}(x^2+y)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2+y}}$ when $x=2, y=3$: $\frac{2}{\sqrt{7}}$

$\frac{\partial z}{\partial y} = \frac{1}{2}(x^2+y)^{-1/2} = \frac{1}{2\sqrt{x^2+y}}$ when $x=2, y=3$: $\frac{1}{2\sqrt{7}}$

$$\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y = \frac{2}{\sqrt{7}} \cdot 0.2 + \frac{1}{2\sqrt{7}} \cdot 0.1$$

$$\approx 0.17$$

Ex 2. An ideal gas satisfies $PV = 0.5T$, where P is pressure, V is volume, T is temperature. A scientist measures volume as $2\,m^3$ with an error of $0.1\,m^3$ and temperature as 200 Kelvin with an error of 5 Kelvin. What is the maximum error in the estimated Pressure? (Round to 2 decimals)

$P = 0.5TV^{-1}$ so $\frac{\partial P}{\partial T} = 0.5V^{-1} = \frac{0.5}{V}$, $\frac{\partial P}{\partial V} = -0.5TV^{-2} = \frac{-0.5T}{V^2}$

$T_{measured} = 200$, $V_{measured} = 2$, $\Delta T = \pm 5$, $\Delta V = \pm 0.1$

so

$$\Delta P \approx \frac{0.5}{V}\Delta T + \frac{-0.5T}{V^2}\Delta V = \frac{0.5}{2}(\pm 5) + \frac{-0.5(200)}{2^2}(\pm 0.1)$$

four possibilities: -1.25, -3.75, 1.25, or 3.75

maximal error is when you get $\boxed{3.75}$ (or -3.75) kPa

Ex 3. A soup can with height h cm and radius r cm has volume $V = \pi r^2 h$. Currently, it has $h = 8$ cm and $r = 3$ cm. If they want to decrease the height by 0.1 cm and keep the volume the same, use differentials to estimate how much they should change the radius. (Round to 3 decimals)

$\frac{\partial V}{\partial r} = 2\pi r h$, $\frac{\partial V}{\partial h} = \pi r^2$, $r_{initial} = 3$, $h_{initial} = 8$, $\underset{\text{(decrease)}}{\Delta h = -0.1}$, $\Delta V = 0$

Find Δr.

$$\Delta V \approx 2\pi r h \cdot \Delta r + \pi r^2 \cdot \Delta h$$

$$0 \approx 2\pi(3)(8)\,\Delta r + \pi(3)^2(-0.1)$$

$$0 \approx 48\pi\,\Delta r - 0.9\pi$$

$$\Delta r \approx \frac{0.9\pi}{48\pi} \approx 0.019$$

Since Δr is positive, $\boxed{\text{increase}}$ radius by about $\boxed{0.019\text{ cm}}$

Ex 4. A company has $P(x,y) = 20x^{3/4}y^{1/4}$ thousand units produced where x is number of employees and y is expenditures in thousands of dollars. Suppose they wish to reduce the employees from 100 to 90 and increase expenditures from \$15,000 to \$18,000.

(a) Estimate the change in productivity due to change in employees.

Want ΔP wrt $x \approx \frac{\partial P}{\partial x}\Delta x$, $\Delta x = 90-100 = -10$

$\frac{\partial P}{\partial x} = 15x^{-1/4}y^{-1/4}$ so ΔP wrt $x \approx 15\cdot(100)^{-1/4}\cdot(15)^{1/4}\cdot(-10)$

≈ -93.350 thousand units

So a <u>decrease</u> of 93.350 thousand units

(b) Estimate the change in productivity due to change in expenditures.

Want ΔP wrt $y \approx \frac{\partial P}{\partial y}\Delta y$, $\Delta y = 18-15 = 3$

$\frac{\partial P}{\partial y} = 5x^{3/4}y^{-3/4}$ so ΔP wrt $y \approx 5\cdot(100)^{3/4}\cdot(15)^{-3/4}\cdot(3)$

≈ 62.233 thousand units

So an <u>increase</u> of 62.233 thousand units

(c) Estimate the total change in productivity.

$\Delta P = \Delta P$ wrt $x + \Delta P$ wrt $y \approx -93.350 + 62.233 \approx -31.117$

So a <u>decrease</u> of 31.12 thousand units

Ex 5. A can of height h cm and radius r cm is in production. The materials for the can cost 0.002 cents per cm^2 and the liquid inside costs 0.001 cents per cm^3. Estimate the change in cost by increasing height by 0.2 cm and decreasing radius by 0.3 cm if $r = 4$ cm and $h = 9$ cm.

$SA = 2\pi r^2 + 2\pi rh$, $V = \pi r^2 h$

so $C = 0.002(2\pi r^2 + 2\pi rh) + 0.001(\pi r^2 h)$

$\frac{\partial C}{\partial r} = 0.002(4\pi r + 2\pi h) + 0.001(2\pi rh)$

$\frac{\partial C}{\partial h} = 0.002(2\pi r) + 0.001(\pi r^2)$

$\Delta C \approx [0.002(2\pi r + 2\pi h) + 0.001(\pi r^2 h)]\Delta r + [0.002(2\pi r) + 0.001(\pi r^2)]\Delta h$

$\approx [0.002(2\pi(4) + 2\pi(9)) + 0.001(\pi(4)^2(9))](-0.3) + [0.002(2\pi(4)) + 0.001(\pi(4)^2)](0.2)$

≈ -0.112, so approx $\boxed{0.112 \text{ cent decrease}}$

MA 16020
Lesson 21

Relative percent error of f at c is

$$\left(\frac{\text{max error of } f}{f(c)} \cdot 100\right) \%$$

Ex 6. If $S = \frac{A}{A-W}$, A is measured to be 2.8 with max error of 0.01, W is measured to be 2.2 with max error of 0.05, approximate the relative percentage error of S.

$$\frac{\partial S}{\partial A} = \frac{(A-W)(1)-(A)(1)}{(A-W)^2} = \frac{-W}{(A-W)^2}$$

$$\frac{\partial S}{\partial W} = \frac{\partial}{\partial W}\left(A(A-W)^{-1}\right) = -A(A-W)^{-2}(-1) = \frac{A}{(A-W)^2}$$

so

$$\Delta S \approx \frac{-W}{(A-W)^2}\Delta A + \frac{A}{(A-W)^2}\Delta W$$

$$\text{error of } S \approx \frac{-2.2}{(2.8-2.2)^2}(\pm 0.01) + \frac{2.8}{(2.8-2.2)^2}(\pm 0.05)$$

$$\approx \pm 0.328 \text{ or } \pm 0.45$$

max error is 0.45

$$\text{Calculate } S = \frac{A}{A-W} = \frac{2.8}{2.8-2.2} = \frac{14}{3}$$

$$\text{relative percent error} = \left(\frac{0.45}{\left(\frac{14}{3}\right)} \cdot 100\right) \% \approx \boxed{9.64\%}$$