MA 16020

Lesson 22

The Multivariate Chain Rule

From last lesson, we know $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$.
If y and x are functions of t, then so is z.
Divide both sides by dt to get

The Multivariate Chain Rule

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$\frac{dz}{dt}$ is the total derivative of z with respect to t.

Ex 1. Use the Chain Rule to find $\frac{dz}{dt}$ at $t=1$
when $z = \sin(xy)$, $x(t) = 3t^2$, $y(t) = 4t$.
(Round to 3 decimals.)

$\frac{\partial z}{\partial x} = \cos(xy)\cdot y$, $\frac{\partial z}{\partial y} = \cos(xy)\cdot x$, $\frac{dx}{dt} = 6t$, $\frac{dy}{dt} = 4$

$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (y\cos(xy))(6t) + (x\cos(xy))(4)$

$x(1) = 3(1)^2 = 3$, $y(1) = 4(1) = 4$

$$\left.\frac{dz}{dt}\right|_{t=1} = (4)\cos(3\cdot 4)\cdot 6(1) + (3)\cos(3\cdot 4)\cdot 4$$
$$= 24\cos(12) + 12\cos(12) \approx \boxed{30.379}$$

Ex 2. At a factory, when capital expenditures are K thousand dollars and L worker-hours of labor are employed, the daily output is $Q(K,L) = 48K^{1/2}L^{1/3}$ units. Currently, expenditures are \$500,000 and increasing at a rate of \$3,000 per day, while 1,000 worker-hours are employed and decreasing at a rate of 3 worker-hours per day. At what rate is production changing? (2 decimals)

Need $\frac{dQ}{dt}$. $\frac{\partial Q}{\partial K} = 24K^{-1/2}L^{1/3}$, $\frac{\partial Q}{\partial L} = 16K^{1/2}L^{-2/3}$

$K = 500$ (K measured in thousands), $L = 1000$

$\frac{dK}{dt} = 3$, $\frac{dL}{dt} = -3$ (decreasing)

$$\frac{dQ}{dt} = (24K^{-1/2}L^{1/3})(3) + (16K^{1/2}L^{-2/3})(-3)$$
$$= 24(500)^{-1/2}(1000)^{1/3}(3) + 16(500)^{1/2}(1000)^{-2/3}(-3)$$
$$\approx 21.47$$

So production is increasing at a rate of $\approx \boxed{21.47 \text{ units/day}}$

Ex 3. The radius of a right circular cylinder is increasing at a rate of 2.5 inches per minute and the height is decreasing at a rate of 12 inches per minute. At what rate is the surface area changing if the radius is 10 inches and the height is 18 inches? (Round to 1 decimal)

$S = 2\pi r^2 + 2\pi rh$, $\frac{\partial S}{\partial r} = 4\pi r + 2\pi h$, $\frac{\partial S}{\partial h} = 2\pi r$

$r = 10$, $h = 18$, $\frac{dr}{dt} = 2.5$, $\frac{dh}{dt} = -10$

$$\frac{dS}{dt} = (4\pi r + 2\pi h)(2.5) + (2\pi r)(-10)$$
$$= (4\pi(10) + 2\pi(18))(2.5) + (2\pi(10))(-10)$$

≈ -31.4, so surface area is decreasing at a rate of $\approx \boxed{31.4 \text{ in}^2/\text{min}}$

Ex 4. The ideal gas law states that $PV = nRT$, where P is pressure, V is volume, T is temperature, and n and R are constants. Find a function for the rate of change of pressure if temperature is decreasing 3 kelvin per minute and volume is increasing at 4 L per minute.

$$P = \frac{nRT}{V} = nRTV^{-1}$$

Find $\frac{dP}{dt}$.

$$\frac{\partial P}{\partial T} = nRV^{-1} = \frac{nR}{V}, \quad \frac{\partial P}{\partial V} = -nRTV^{-2} = -\frac{nRT}{V^2}$$

$$\frac{dT}{dt} = -3, \quad \frac{dV}{dt} = 4$$

$$\frac{dP}{dt} = \left(\frac{nR}{V}\right)(-3) + \left(-\frac{nRT}{V^2}\right)(4)$$

$$= \boxed{-\frac{3nR}{V} - \frac{4nRT}{V^2} \text{ kPa/min}}$$