MA 16020
Lesson 23
Relative Extrema (Part 1)

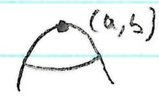
Recall: Given a univariate function $y = f(x)$,
you have the second derivative test.
Find critical #s by setting $f'(x) = 0$.
Find $f''(x)$, and then plug in the critical #s.
If $f''(c) > 0$, get local min. If $f''(c) < 0$, get local max.
If $f''(c) = 0$, test is inconclusive.
We get a similar test for $z = f(x, y)$

Second Derivative Test

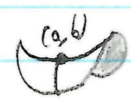
Given $z = f(x,y)$

1. Find critical points by finding pairs (a,b) which satisfy both $f_x(a,b) = 0$ and $f_y(a,b) = 0$ simultaneously
2. Find f_{xx}, f_{yy}, and f_{xy}.
3. Compute the sign of the discriminant function $D(a,b) = f_{xx}(a,b) \cdot f_{yy}(a,b) - (f_{xy}(a,b))^2$ at each critical point (a,b)
4. a) If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then f has a local min at (a,b)

   b) If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then f has a local max at (a,b)



   c) If $D(a,b) < 0$, then f has a saddle point at (a,b)



   d) If $D(a,b) = 0$, then the test is inconclusive.

Note: For 4a or 4b, you can use $f_{yy}(a,b)$ instead of $f_{xx}(a,b)$ if you wish. In order for $D(a,b)$ to be positive, $f_{xx}(a,b)$ and $f_{yy}(a,b)$ must be the same sign.

Why does this work?

If $D(a,b) > 0$, then $f_{xx}(a,b)$ and $f_{yy}(a,b)$ have the same sign, meaning f is either concave up in both x- and y-directions (giving a local min) or is concave down in x- and y-directions (giving a local max).

If $D(a,b) < 0$, then f has a max in one direction and a min in the other (giving a saddle point)

Ex 1. Find and classify all critical points of
$f(x,y) = 3x^2 - xy + 6y^2 - 12x + 2y + 3$

$f_x = 6x - y - 12$, $f_y = -x + 12y + 2$

Set $f_x = 0$ and $f_y = 0$ to get critical points

$6x - y - 12 = 0$, $-x + 12y + 2 = 0$

Solve system of equations

$y = 6x - 12$ plug in: $-x + 12(6x-12) + 2 = 0$

$71x - 142 = 0 \Rightarrow x = 2$

When $x = 2$, $y = 6(2) - 12 = 0$

critical point: $(2, 0)$

$f_{xx} = 6$, $f_{yy} = 12$, $f_{xy} = -1$

$D(x,y) = (6)(12) - (-1)^2 > 0$ for all points (a,b)

and $f_{xx}(2,0) = 6 > 0$ so

f has a $\boxed{\text{local minimum at } (2,0)}$

Ex 2. Find and classify the critical points for $g(x,y)$ where $g_x = -3x + y$ and $g_y = x - \frac{1}{243}y^3$

Set $g_x = 0$ and $g_y = 0$

$-3x + y = 0$ and $x - \frac{1}{243}y^3 = 0$

$y = 3x$ plug in: $x - \frac{1}{243}(3x)^3 = 0 \Rightarrow x - \frac{1}{9}x^3 = 0$

$\Rightarrow x\left(1 - \frac{1}{9}x^2\right) = 0$

get $x = 0$ or $x = 3$ or $x = -3$

$x = 0$: $-3(0) + y = 0 \Rightarrow y = 0$ $(0,0)$

$x = 3$: $-3(3) + y = 0 \Rightarrow -9 + y = 0 \Rightarrow y = 9$ $(3,9)$

$x = -3$: $-3(-3) + y = 0 \Rightarrow 9 + y = 0 \Rightarrow y = -9$ $(-3,-9)$

$g_{xx} = -3$, $g_{yy} = -\frac{1}{81}y^2$, $g_{xy} = 1$

$D(x,y) = (-3)\left(-\frac{1}{81}y^2\right) - (1)^2 = \frac{1}{27}y^2 - 1$

$D(0,0) = \frac{1}{27}(0)^2 - 1 = -1 < 0$ $\boxed{\text{saddle point at } (0,0)}$

$D(3,9) = \frac{1}{27}(9)^2 - 1 = 3 - 1 > 0$, $g_{xx} = -3 < 0$, $\boxed{\text{local max at } (3,9)}$

$D(-3,-9) = \frac{1}{27}(-9)^2 - 1 > 0$, $g_{xx} < 0$, $\boxed{\text{local max at } (-3,-9)}$

Ex 3. Find and classify the critical points of
$h(u,v) = 3u^2v + 48uv + 4v^2$

$h_u = 6uv + 48v$, $h_v = 3u^2 + 48u + 8v$

set $h_u = 0$ and $h_v = 0$

$6uv + 48v = 0 \Rightarrow v(6u+48) = 0 \Rightarrow v = 0$ or $6u + 48 = 0 \Rightarrow u = -8$

Plug into other equation.

$v = 0$: $3u^2 + 48u + 8(0) = 0 \Rightarrow 3u^2 + 48u = 0$

$3u(u+16) = 0 \Rightarrow u = 0$ or $u = -16$

(u,v) are $(0,0)$ and $(-16,0)$

$u = -8$: $3(-8)^2 + 48(-8) + 8v = 0 \Rightarrow -192 + 8v = 0 \Rightarrow v = 24$

$(-8, 24)$

$h_{uu} = 6v$, $h_{vv} = 8$, $h_{uv} = 6u + 48$

$D(u,v) = (6v)(8) - (6u+48)^2$

$D(0,0) = (6(0))(8) - (6(0)+48)^2 < 0$ $\boxed{\text{Saddle point at } (0,0)}$

$D(-16,0) = (6(0))(8) - (6(-16)+48)^2 < 0$ $\boxed{\text{Saddle point at } (-16,0)}$

$D(-8,24) = (6(24))(8) - (6(-8)+48)^2 = 1152 - 0 > 0$

$h_{uu}(-8,24) = 6(24) > 0$ $\boxed{\text{local min at } (-8,24)}$